EXISTENCE THEOREMS

FOR FLOORPLANS

A Thesis

presented for the degree

of

Doctor of Philosophy

at the

University of Canterbury

Christchurch

New Zealand

by

I. Rinsma

1987

CONTENTS

ACKNOWLEDGEMENTS

Many people have contributed either directly or indirectly towards the production of this thesis. My main debt of gratitude is to my supervisor, Dr David Robinson, who introduced me to the topic. Without his continuing enthusiasm, guidance and constructive criticism this thesis would never have been completed. Special thanks also go to Dr Derrick Breach who was my acting supervisor whilst Dr Robinson was on leave. I am also very grateful to Dr Brendan McKay and Dr Nick Wormald for their helpful suggestions regarding the proof of theorem 6.4, and Dr John Hannah for his assistance in proof reading. Gill Evans is also to be thanked for her excellent typing.

Finally I wish to thank the Mathematics Department of the University of Canterbury and the University Grants Committee who provided financial support.

ABSTRACT

The existence of floorplans with given areas and adjacencies for the rooms cannot always be guaranteed. Rectangular, isometric and convex floorplans are considered. For each, the areas of the rooms and a graph representing the required internal adjacencies between the rooms is given. This thesis gives existence theorems for a floorplan satisfying these conditions. If the graph is maximal outerplanar, only a convex floorplan can always be guaranteed. Floorplans of each type can be found if the graph is a tree.

A branching index is defined for a tree, and used to give the minimum number of vertices of degree 2 in any maximal outerplanar graph, in which the tree can be embedded.

If the graph of adjacencies is a tree, and each room in the plan is external, once again only convex floorplans can always be guaranteed. Rectangular floorplans can always be found in some cases, depending on the embedding index of the tree.

CHAPTER I

INTRODUCTION

An area of research common to architectural design and facility layouts is the designing of planar floorplans, composed of nonoverlapping rooms divided from each other by walls, to suit given topological and dimensional constraints.

Topological constraints are usually adjacencies between rooms and with the exterior of the plan. Interconnections between rooms and with the exterior, and natural lighting or ventilation into the rooms, are often reasons for such constraints. Dimensional constraints involve shapes or sizes of each room and the actual floorplan - for example, rectangular or convex rooms with certain areas, proportions, or lengths of walls.

These constraints interact limiting the choice of feasible solutions. If a plan is to correspond to a rectangular dissection, say, then there are certain limitations on the number and type of adjacencies the overall plan can have. Further, even if a number of solutions exist, they may be difficult to find due to the combinatorial nature of the problem.

The use of computers for automated floorplan design using either heuristics or exhaustive methods is outlined in chapter II. Graph theoretic approaches have been used over the past twenty years. Most concentration has been on rectangular floorplans.

In the case of area constraints the equations to be solved for a particular rectangular floorplan can be found (Earl and March (1979)) but a solution may not always satisfy the adjacency requirements. On the other hand, very few of the topologically feasible floorplans may yield feasible solutions in the dimensioning step. It seems that, at this point in the investigation, it was unknown whether or not a rectangular floorplan could always be found to satisfy both given adjacency and area requirements.

This thesis studies this problem. The emphasis is on existence theorems. A graph theoretic approach is used.

I. PROBLEMS TO BE STUDIED

We concentrate mainly on plans in which each room is adjacent to the exterior, and on three types of floorplans. These are defined as rectangular, isometric and convex. The adjacency requirements are represented in a graph. If each room is external, the graph must be outerplanar (Lynes (1977)), and the allowable graphs range from trees to maximal outerplanar graphs.

For each type of plan we investigate whether a plan can be found to satisfy the areas and adjacency requirements represented either by a tree or maximal outerplanar graph.

II. ORGANISATION OF THESIS

In chapter II preliminary definitions and explanations are given, along with a review of the current literature.

Chapter III is solely graph theoretic. Properties of trees and outerplanar graphs are given. A new index is defined for a tree. This is used to give restrictions on the types of maximal outerplanar graphs any given tree can be embedded in.

Chapter IV considers existence theorems for rectangular floorplans having area and adjacency constraints given by a maximal outerplanar graph. These are then extended to the isometric and convex floorplans in chapter V. In chapters VI and VII the adjacency constraints are instead given by a tree. It is shown that at least one floorplan of each of the three types can then be found to satisfy any given area constraints. This is seen not to be the case in chapter VII if the further condition that each room is adjacent to the exterior is imposed.

The thesis ends with a review of the study, possible applications and extensions for future research.

CHAPTER II

REVIEW OF THE LITERATURE

This chapter reviews the work that has been done on designing floorplans, and the various techniques used to represent the plans. First some preliminary definitions and comments are given. Unless otherwise stated the terminology and notation of Harary (1969) in relation to graphs is used.

I. PRELIMINARY DEFINITIONS AND EXPLANATORY REMARKS

Definition 2.1 A floorplan is a polygon, the *plan boundary*, divided by straight lines into component polygons called the *rooms*. The edges forming the perimeter of each room are called *walls*. The region not enclosed by the boundary is called the *exterior*.

Definition 2.2 A point in a floorplan where three or more walls coincide is called a *joint*. Joints are further classified as n-joints where n is the number of walls that meet at that point.

Definition 2.3 A continuous length of wall between two joints is called a *wall section*. This can be either an *external wall section*, forming part of the plan boundary or an *internal wall section*.

To each floorplan there corresponds a number of graphs.

Definition 2.4 Let every joint in the plan be represented by a vertex, and let an edge exist between two vertices whenever a section of wall in the floorplan runs between the corresponding joints. The resulting graph is known as the *plan graph* (Steadman (1983)).

<u>Definition 2.5</u> Two rooms in the floorplan are *adjacent* if they share some wall section. It is not sufficient for them to touch at a point only. Similarly a room is adjacent to the exterior if it has a wall section in common with the plan boundary.

<u>Definition 2.6</u> To each floorplan corresponds an *adjacency graph* in which the vertices represent the rooms, and the exterior, and two vertices are joined by an edge whenever there is a wall section in the plan common to both the corresponding regions.

Figure 2.1 shows three different floorplans (a), each with the same plan graph (b) and adjacency graph (c). In each B,E,H,K,L, and M are joints. In (a)(i) room a has walls IK, KH and HI, while the wall sections in the floorplan are KAB, BCDE, EFGH, HIK, HL, LK, LM, BM and EM.

The plan graph is a diagrammatic version of the floorplan. For example, if in figure 2.1(b) we imagine the edges of the plan to be elastic bands, then it can be "stretched" to form any one of the floorplans in figure 2.1(a).

The plan graph and adjacency graph have a special relationship to each other - the adjacency graph is the dual of the plan graph and viceversa. That is, for each vertex in one graph there corresponds a face in the other.

From the way in which the plan graph was defined, it follows that the plan graph is always plane. A plane graph has a plane dual, so a graph must be planar if it is to be the adjacency graph of some plan. Also every adjacency graph is connected.

Either the adjacency graph or plan graph or both can have multiple edges. That is, they are multigraphs.

Figure 2.2 shows a plan (a), with its corresponding plan graph

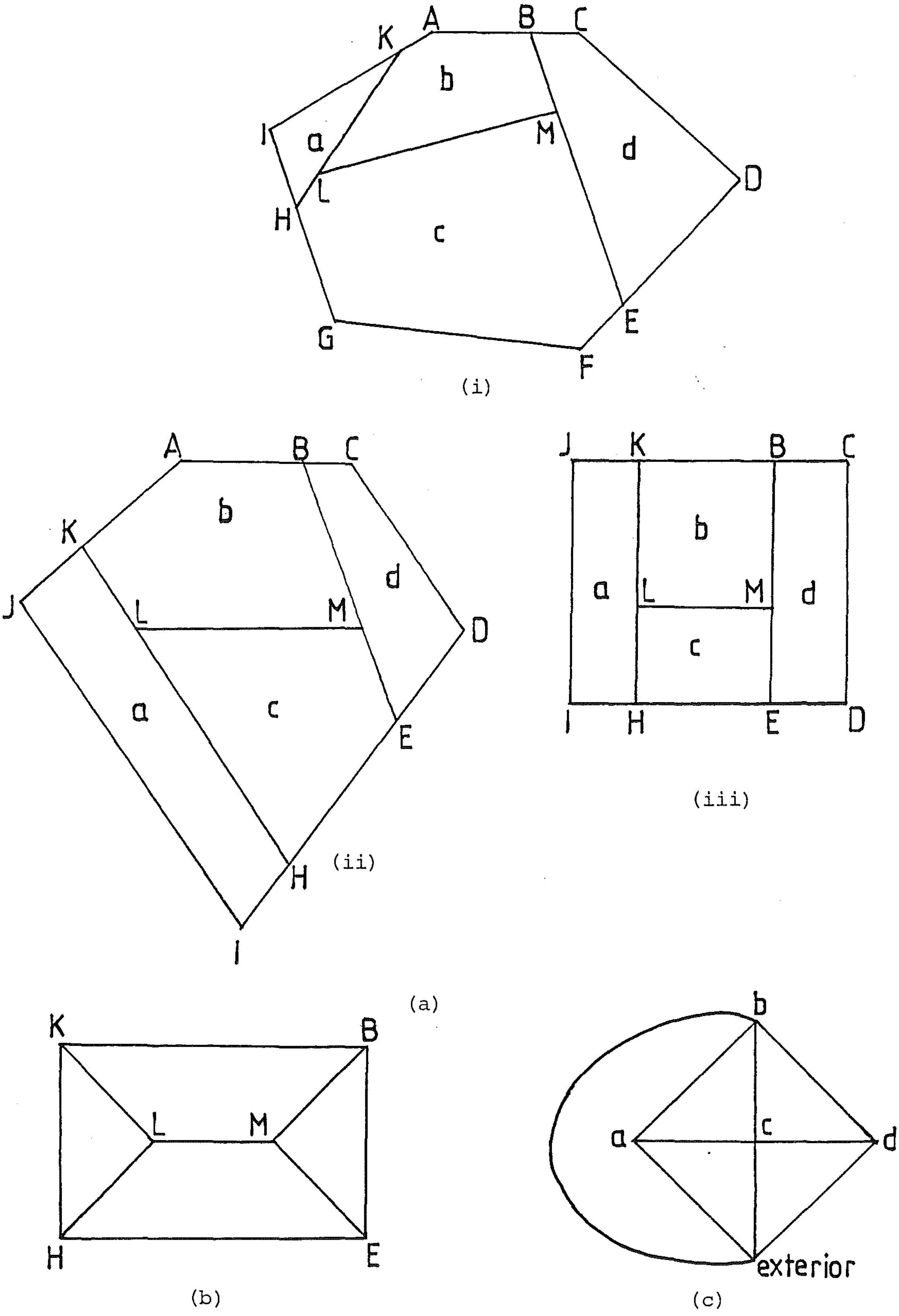
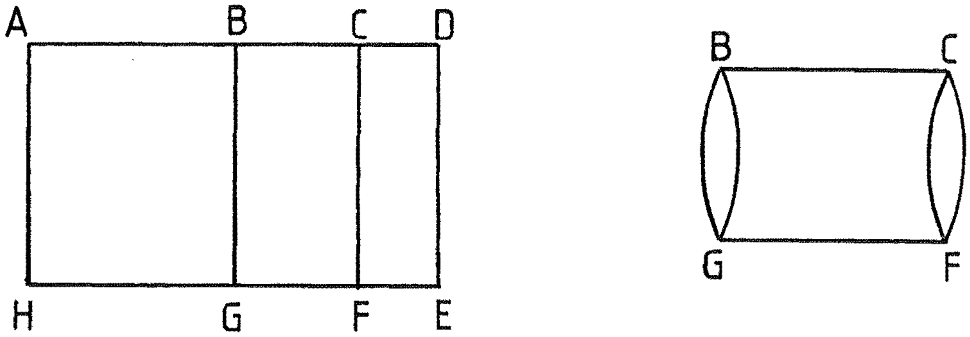


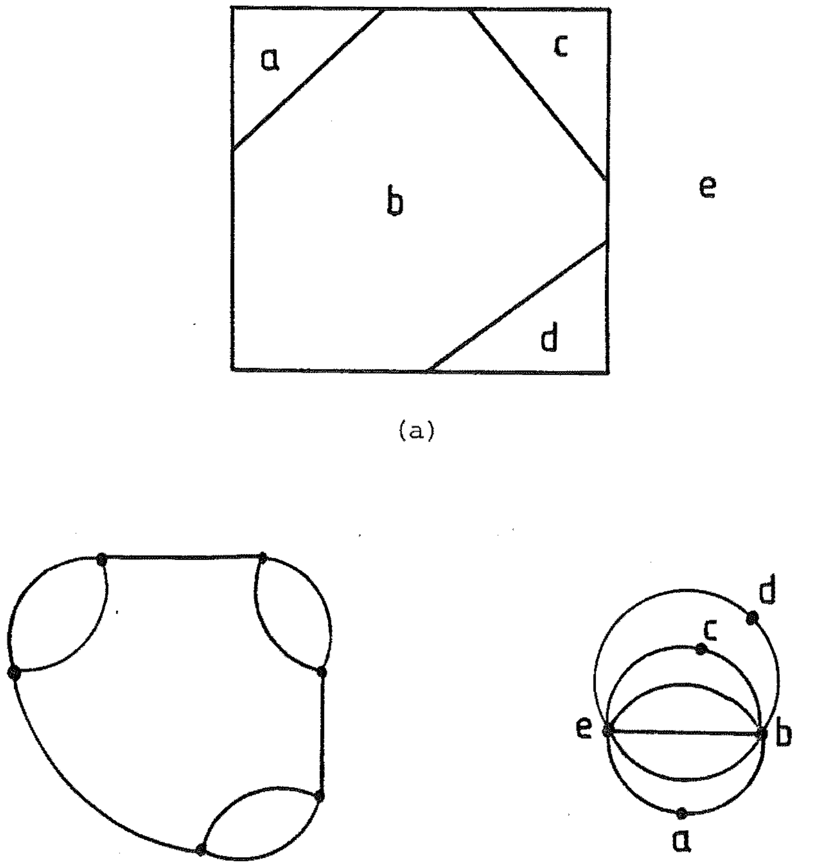
Figure 2.1 Three floorplans (a) with the same plan graph (b) and adjacency graph (c).

which is a multigraph. Figure 2.3 shows a plan (a) for which both the plan graph (b) and adjacency graph (c) are multigraphs.



(a) (b)

Figure 2.2 A floorplan (a) and its plan graph (b).



(a)

(b) (c)

Figure 2.3 A floorplan (a) whose plan graph (b) and adjacency graph (c) are multigraphs.

<u>Definition 2.7</u> A *through room* has two of its walls not in the same wall section, lying on the plan boundary.

Room b in figure 2.3(a) is a through room. Through rooms imply the presence of multiple edges in the adjacency graph.

Definition 2.8 An *external room* has at least one of its walls forming part of the plan boundary, while an *internal room* has none.

Definition 2.9 Another type of adjacency graph is called the *weak dual* by Earl and March (1979). This is the adjacency graph formed as in definition 2.5 above, with the exterior ignored, so each edge represents an adjacency between two rooms in the plan.

Definition 2.10 A *rectangular floorplan* is a floorplan in which the plan boundary and each room are rectangles.

A. Rectangular floorplans

The following definitions and remarks concern only rectangular floorplans.

Definition 2.11 Through rooms and *corner rooms* have exactly two walls forming part of the plan boundary. For corner rooms these walls are adjacent, while for through rooms they are opposite each other.

Definition 2.12 An *endroom* has three walls on the plan boundary.

Definition 2.13 A *wall segment* is a maximum continuous sequence of aligned straight wall sections. If each wall section is internal (external), then it is an *internal* (*external*) *wall segment*.

Definition 2.14 A *fault line* is an internal wall segment, joining points in opposite sides of the plan boundary.

In figure 2.4(a) A is an endroom, B a through room while both C and D are corner rooms. All joints in the plan are 3-joints except α which is a 4-joint. The wall segment from X to Y is a fault line. Rooms E and F are adjacent, but not rooms E and H as they meet only at a 4-joint. Figure 2.4(b) shows that the plan has seventeen internal wall sections and nine external wall sections. Clearly any rectangular floorplan has four external wall segments - the sides of the plan boundary.

Definition 2.15 The four external wall segments of the plan are called the *north*, *south*, *east* and *west sides* (see figure 2.5).

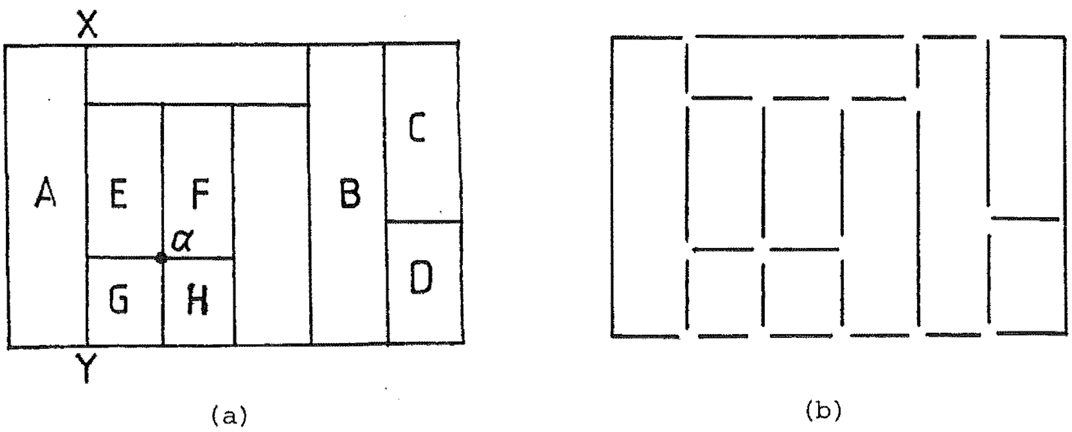


Figure 2.4 A rectangular floorplan (a) with its wall sections (b).

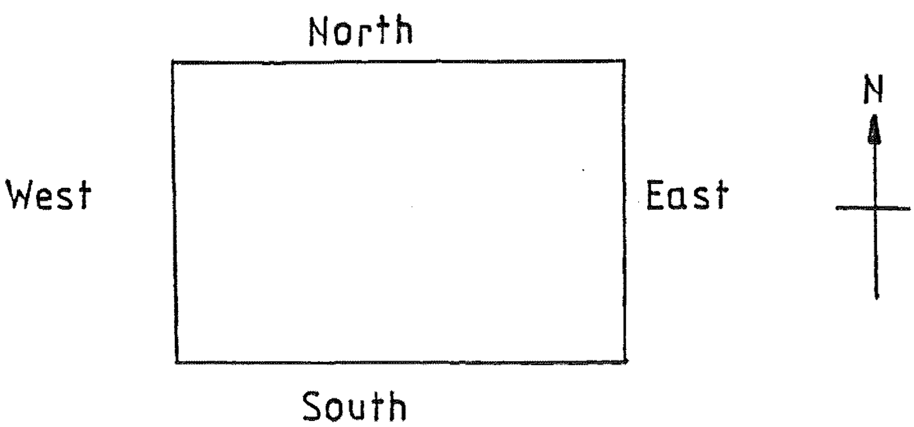


Figure 2.5 The sides of a rectangular floorplan.

Definition 2.16 A *north room* is a room having a wall on the north side. Similar definitions exist for south, east and west rooms. A corner room, like C in figure 2.4 is therefore both a north and east room.

Definition 2.17 For a given floorplan, let the sets V and H consist of those internal wall sections which are parallel to the west or north sides of the plan respectively. Then {V,H} is a partition of the set of internal wall sections in the plan.

Definition 2.18 The floorplan with one room is called *trivial*, and generally will be excluded from the discussion.

B. Properties of rectangular floorplans

The following need no proof.

1. Every rectangular floorplan has at most two endrooms.
2. Every rectangular floorplan has at most four corner rooms.
3. Since walls meet at right angles, only 3-joints and 4-joints are possible in any rectangular floorplan.

II. REVIEW OF PREVIOUS WORK

The remainder of this chapter outlines the research made into the design of floorplans, particularly rectangular, with given topological or dimensional constraints. Various ways of representing floorplans are given.

A. Representing rectangular floorplans

1. Gratings

Mitchell, Steadman and Liggett (1976) introduced minimal gratings, arising from the work of Newman (1964) to describe rectangular floorplans. A coordinate system or grid is imposed on the floorplan, with every grid line corresponding to at least one wall in the floorplan, using the minimum number of grid lines possible to mark the position of all walls.

Consider figure 2.6(a) in which two floorplans have essentially the same 'shape'. In (b) the gratings of each floorplan are shown superimposed on the plan. The dimensions of the minimum gratings can be adjusted so that each cell in the grating becomes square, as shown in (c). This representation is unique and is called the *dimensionless representation* or canonical version of the plan.

It does not alter the topology of the original figure, as rooms adjacent in the plan remain adjacent in the canonical version. Also it is possible to return to any original dimensional floorplan from the dimensionless version with an appropriate set of dimensions, giving the required spacings for the grating in the x and y directions.

The grating size of any grating is given by l x m, where m is at least equal to l. Thus the grating in figure 2.6 has size 2 x 3.

Galle (1986) pointed out that these gratings are essentially the same as the 'rectangular meshes' of Velez-Jahn (1971). He also defined another type of representation - δ the abstraction module, and an operation called δ-derivation for a dimensioned rectangular floorplan. This operation creates an approximate floorplan and is a type of *abstraction*. He investigated its properties and outlined how it could be used in design problems.

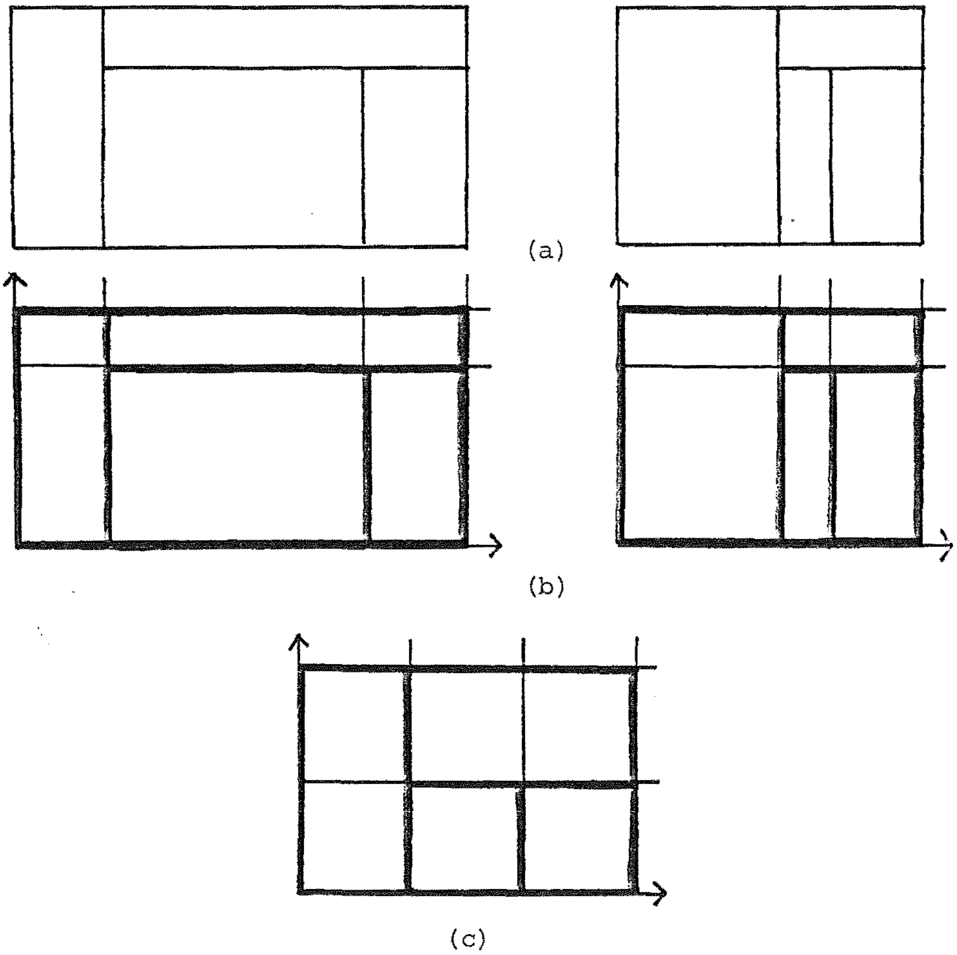


Figure 2.6 Two floorplans (a) with their gratings (b) and identical dimensionless representation (c).

2. Isomorphic floorplans

Definition 2.19 Let F_1 and F_2 be two undimensioned labelled rectangular floorplans whose sets of internal wall sections have partitions $\{V_1,H_1\}$ and $\{V_2,H_2\}$ respectively. Then F_1 is *isomorphic* to F_2 if

i) the rooms in F_1 are in one-to-one correspondence to the rooms in F_2,

ii) two rooms in F_1 are adjacent if and only if the corresponding rooms in F_2 are adjacent. (This induces a one-to-one correspondence between the internal wall sections),

iii) the internal wall sections in V_1 are in one-to-one correspondence to those in V_2, and H_1 to H_2, or V_1 to H_2 and H_1 to V_2.

Here part (iii) deals with isomorphisms of floorplans under either rotation of 90^o, 180^o or 270^o, or reflection in a mirror line parallel to one of the sides, or a combination of both.

Three of the four floorplans in figure 2.7, namely F_1, F_2 and F_3 are isomorphic. F_1 and F_4 satisfy (i) and (ii) above but not (iii). V_1 corresponds to V_2 and H_3, while H_1 corresponds to H_2 and V_3.

Note that the adjacencies of rooms to the exterior is not used in the definition for isomorphism. This is because, as is shown later, once the plane weak dual and sets V and H of a floorplan are known, the rooms adjacent to the exterior, their type and the order in which they occur around the plan boundary can be found.

Note also that isomorphism is an equivalence relation.

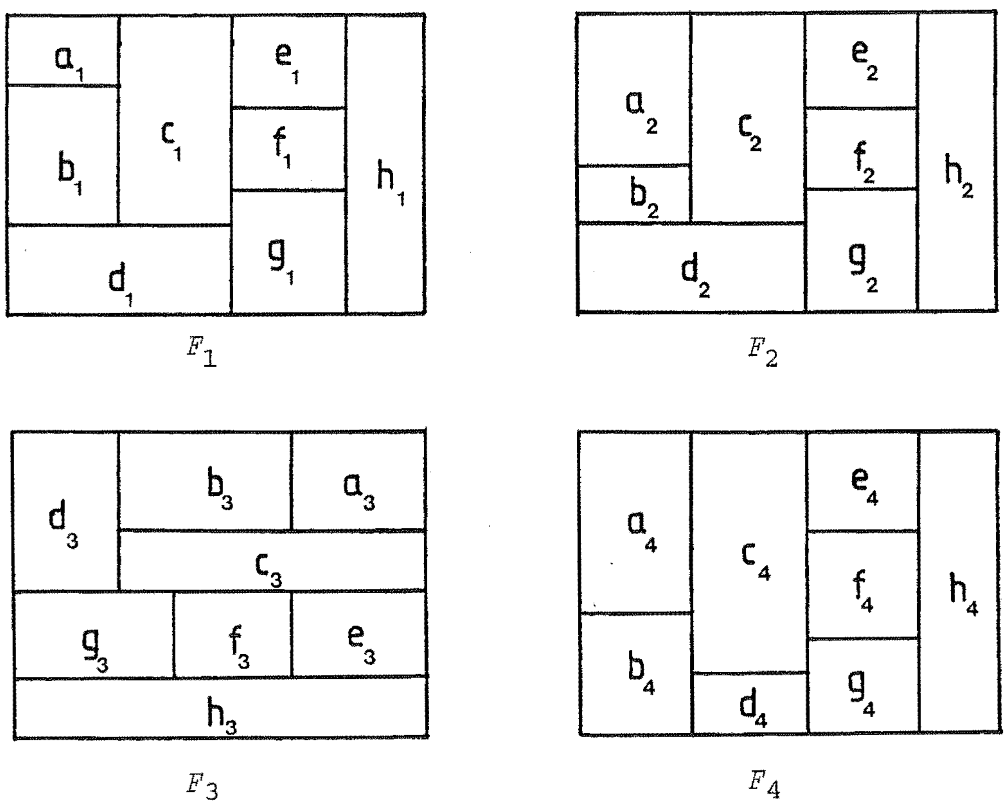


Figure 2.7 Four labelled undimensioned rectangular floorplans with the same weak dual. F_1, F_2 and F_3 are isomorphic.

3. Trivalent and fundamental floorplans

Consider the two floorplans (a) and (b) shown in figure 2.8.

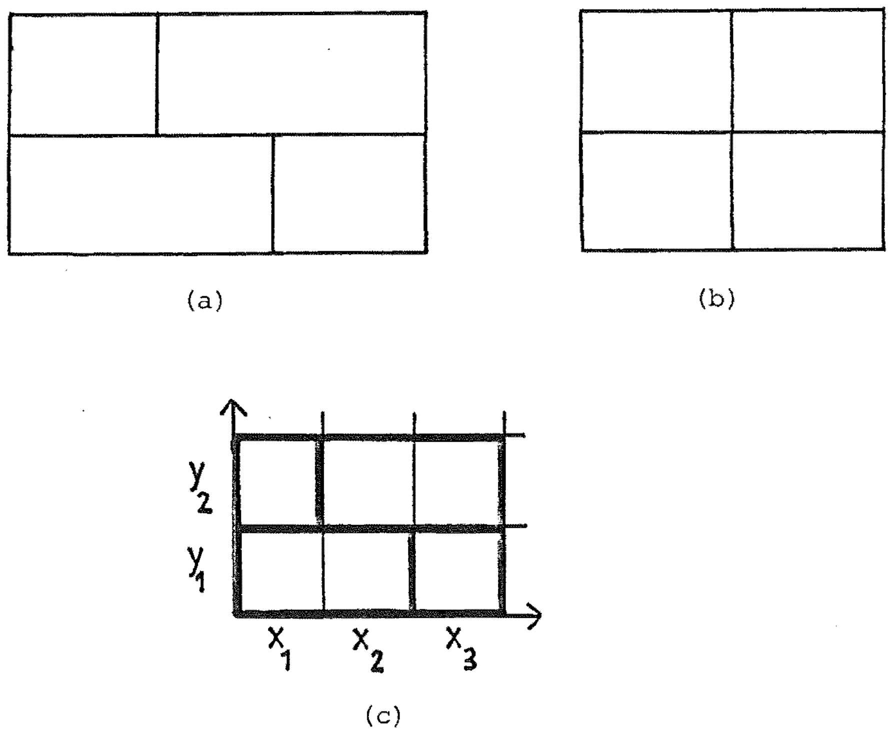


Figure 2.8 Two floorplans (a) and (b) with the grating (c) of (a).

In (b) there is a 4-joint while in (a) none exist. The minimal grating corresponding to (a) is shown in (c) where x_1, x_2, x_3, y_1, y_2 are the dimensioning variables. If x_2 is set equal to zero, then plan (a) is transformed into plan (b). That is the two 3-joints collapse into a single 4-joint, as shown in figure 2.9.

Floorplans in which all joints are 3-joints have been called *trivalent*.

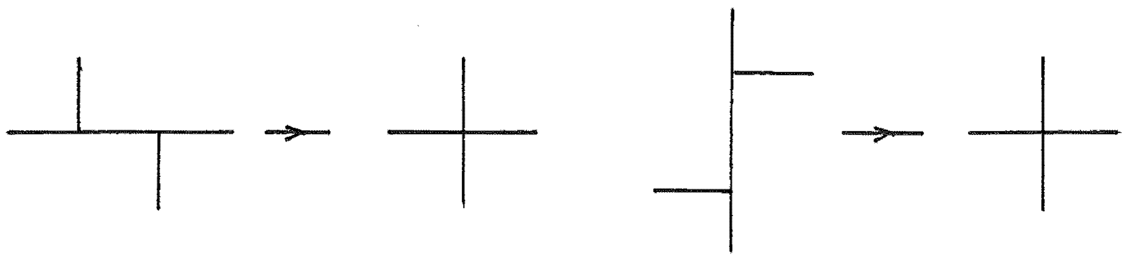


Figure 2.9 Collapse of two 3-joints into a 4-joint.

Another problem with gratings can also occur. Consider the two floorplans shown in figure 2.10(a), and the corresponding gratings in (b). In (ii) two of the internal wall sections are aligned and hence correspond to the same grid line, while in (i) they do not. If y_2 is set equal to zero in (i) then the grating shown on the right is obtained. Thus the 'aligned' floorplan can be treated as a particular type of 'nonaligned floorplan'.

A nonaligned floorplan which is also trivalent has been termed *fundamental*.

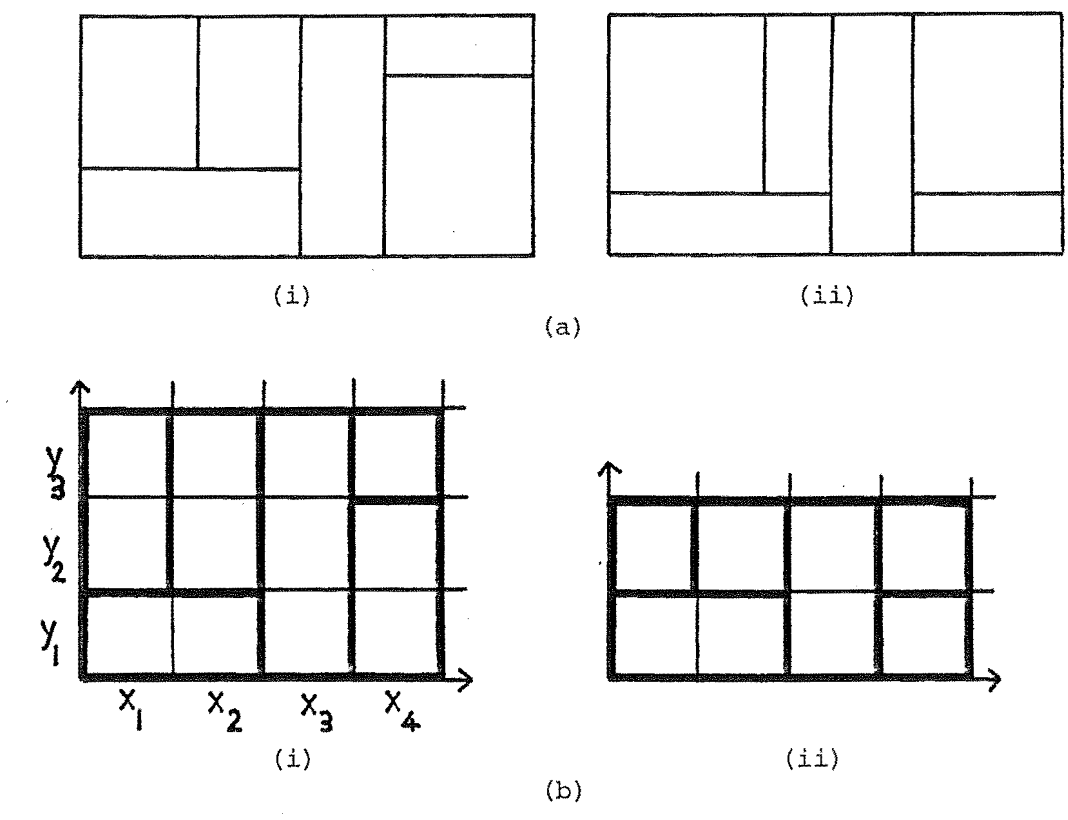


Figure 2.10   Two floorplans (a) with corresponding gratings (b).

B. Generating and counting rectangular floorplans

A rectangular floorplan can be thought of as either the addition of rectangular pieces, like tiles, to produce a rectangular plan, or the division of a large rectangle into smaller rectangular pieces. The latter case has been called a *rectangular dissection*. Many different approaches have been used to enumerate rectangular floorplans.

The first attempt to devise an algorithm to generate rectangular dissections was made by Steadman (1973) using a dissection method. This was not exhaustive. A second algorithm, by Mitchell et al (1976), used three operations, of both addition and dissection type, applied to all dissections with $n-1$ rectangles to generate dissections with n rectangles. This was implemented as a computer program by Sauda (1975) and used to generate dissections up to $n = 8$. However Earl (1977) showed this was not exhaustive for $n = 16$. He devised a new algorithm, claiming it to be exhaustive for all n. His method, unlike the earlier ones, produced only fundamental dissections.

Flemming (1978) developed a new way of describing plans, called *wall representations*, a more general approach than gratings, to provide an exhaustive enumeration of trivalent dissections.

Bloch (1976) predicted theoretically the range of grating sizes needed for all dissections with n rectangles. From this a tiling algorithm was devised:- each feasible grating size for a particular value of n was divided into n rectangular tiles, and then checked to see whether they could be placed together in such a way to fill an empty grating of the predetermined size. From this he was able to generate all dissections up to $n = 19$ (Bloch (1979a, 1979b)).

All dissections up to $n = 7$ were depicted individually in a catalogue, organised to a whole series of classifications. These included the grating sizes, number of external rooms, symmetries, and degrees of the rooms in the weak dual adjacency graph of the plan.

The exact number of dissections in the various categories, with or without alignments or 4-joints were given by Bloch and Krishnamurti (1978) for values up to $n = 10$. These were mainly due to Krishnamurti (Krishnamurti and Roe (1978), Bloch and Krishnamurti (1978)) who designed another method of generating dissections, essentially by assigning imaginary colours to the grating cells. Four rules were given, and the various classes of dissections mentioned above, were generated using different combinations of the rules. A minimum colouring was used to ensure only nonisomorphic rectangular floorplans were produced, making it an extremely fast algorithm.

Another categorisation of rectangular floorplans by Combes (1976) gave properties of a graph relating the number of external walls to internal walls, *partitions*, for each rectangular dissection with n rectangles. He also demonstrated various general relationships existing between the number of external and internal walls, and the number of 3-joints and 4-joints, and n, for a rectangular dissection. These relationships were shown later by Gutiérrez (1979) to be derivable from Euler's polyhedral formula, and graph theory.

As will be shown in Section D, graphs have also been used to represent and enumerate floorplans.

An objection by Stiny (1979) to the work so far was that rectangular dissections represented a restricted class of designs.

C. Nonrectangular plans

March, Matela and O'Hare have done similar work on polyominoes. March and Matela (1974) categorised polyominoes in a way similar to Combes, while Matela and O'Hare (1976) investigated some of the relationships between polyominoes and their weak dual adjacency graphs.

There are other related floorplans, for example, rectangular rooms with a non rectangular boundary, or plans with L, U and + shape rooms, related to rectangular floorplans. (Stiny and Mitchell (1978)). These can be represented by rectangular floorplans, by subdivision or adding 'dummy' rooms. (Steadman (1983)).

Earl (1980) proposed a classification to include all architectural arrangements whose walls are along one or other of two perpendicular directions. This was based on the nature of endpoints of walls in an arrangement. He showed that many of the graph-theoretic representations of rectangular floorplans, and those forms used by Flemming (1978) and Mitchell et al (1976), were related at a more general level. He used shape grammars, developed by Stiny (1975) and Gips (1975) as a special design language involving Boolean operators and description functions specifying how various two-dimensional and three-dimensional shapes may be assembled together, to construct his classes of shapes.

There are many classes of regular nonrectangular floorplans, for example, plane tessellations in which tiles of one or more different shapes are packed together in repeating patterns to fill the plane (Krishnamurti and Roe (1979)), or triangular and hexagonal analogues of polyominoes (Lunnon (1972)).

However, studies of actual building (Bemis (1936), Krüger (1979)) have shown these types are rather rare. Krüger, for example, studied all the buildings in the city of Reading, and found 98% of them to be of rectangular geometry.

Both Krishnamurti (1979) and Earl (1978) have extended their enumerations to the third dimension, that is, to packing rectangular blocks within a rectangular box.

D. Graphs of floorplans

A large amount of literature concerns the application of graph theory to floorplans and architecture. General accounts of architectural applications can be found in March and Steadman (1971, chapters 10 and 11), Steadman (1973) and Earl and March (1979). We now review the ideas and results most relevant to this thesis.

1. Types of graphs

As mentioned in Section I of this chapter, the floorplan can be described as a *plan graph*, in which the vertices are the joints, and the edges wall sections between joints, of the floorplan. This diagrammatic version is similar to the bubble diagrams used by Korf (1977).

The *adjacency graph* and the *weak dual* have vertices representing regions, and edges adjacent regions of the floorplan. The adjacency graph is the dual of the plan graph, unlike the weak dual which is concerned only with internal adjacencies between rooms. Thus there is a vertex in the weak dual for every room in the floorplan, an edge for every internal wall section, and a face for every internal joint.

As the plan graph is planar, every adjacency graph and weak dual is planar. Furthermore the two types of adjacency graphs are connected. If the rectangular floorplan has no through rooms, the adjacency graph has no multiple edges or loops, and the weak dual has no cut vertices.

Adjacency relationships are important, for whenever two rooms share a sufficient length of wall, then it is possible for them to be made accessible to each other via a door. Also overall patterns of adjacency determine circulation routes for a building. Further, rooms having adjacencies to the exterior can have windows thus providing natural lighting, and ventilation.

In fact, Levin (1964) and Hillier and Hanson (1984) used an *access graph* in which the vertices represented the rooms or exterior of a floorplan, and each edge the existence of a door or means of access between two regions.

Earl and March (1979) used a further type of adjacency graph, termed the *augmented dual* for a rectangular floorplan. Here the exterior was divided into four regions - the 'north', 'east', 'south' and 'west' sides - separated by four infinite edges attached to the vertices at the corners of the floorplan. Adjacencies of rooms to these regions and the adjacencies between the four regions themselves were added to the weak dual.

Figure 2.11 shows a rectangular floorplan (a) with its plan graph (b), adjacency graph (c), weak dual (d) and augmented dual (e). Rooms b and d are not adjacent as they meet only in a point.

2. Colouring an adjacency graph

Many floorplans can have the same weak dual, as is shown by figure 2.12.

The edges of any adjacency graph or weak dual of any rectangular floorplan can be coloured in either of two 'colours' to specify the directions in which the corresponding walls lie. Reflecting a given rectangular floorplan in a line parallel to one or other of the sides of the plan, or rotating it 180^0 or 360^0 does not alter the directions in which the walls lie. However, rotating it 90^0 or 270^0 makes all north-south walls lie east-west and viceversa. Thus two labelled rectangular floorplans F_1 and F_2 are isomorphic if either their corresponding coloured adjacency graphs (or weak duals) are identical, or each edge in the coloured adjacency graph (or weak dual) of plan F_1 is coloured differently from the corresponding coloured edge in the adjacency graph (or weak dual) of plan F_2.

This colouring must obey specific rules, if the corresponding plan is a rectangular floorplan (Grason (1968), Earl and March (1979)).

Let the four corners of the floorplan correspond to the four vertices A,B,C,D appearing in cyclic order around the exterior face of the floorplan's weak dual. Note these vertices may not be distinct:- an endroom in the floorplan corresponds to two consecutive corner vertices.

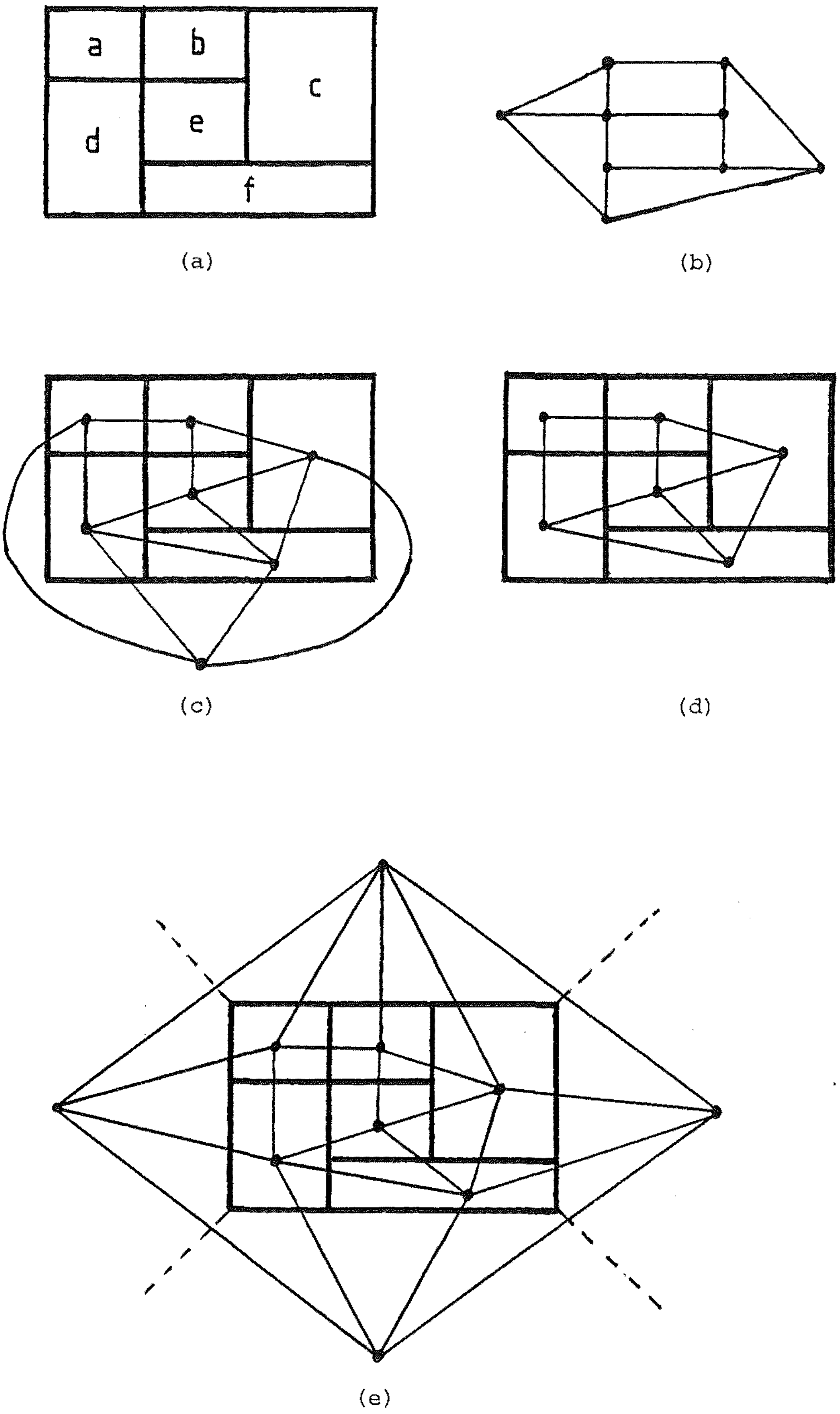


Figure 2.11 The various graphs associated with floorplans:- plan graph (b), adjacency graph (c), weak dual (d) and augmented dual (e) of floorplan (a).

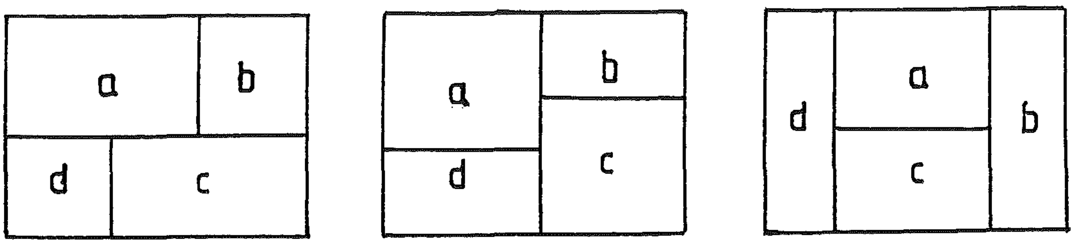
The following four rules are required to ensure a colouring of a weak dual, with two colours, (red and blue, say) corresponds to a rectangular floorplan:-

(i) adjacent edges round the exterior face of the weak dual between A and B, and C and D are coloured red, while those between B and C, and D and A are coloured blue;

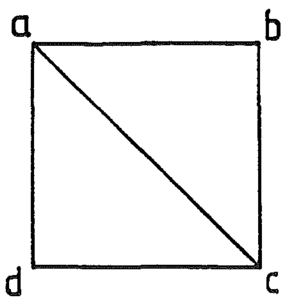
(ii) no triangular face has edges all of one colour;

(iii) adjacent edges round any face with 4 edges, except the exterior, must be coloured differently,

(iv) the edges around any vertex can be partitioned into k groups of consecutive edges of the same colour, where k is 4 for an internal room, 2 for a corner room, 1 for an endroom (or for a through room if they are permitted) and 3 for every other external room.



(a)



(b)

Figure 2.12 Three nonisomorphic floorplans (a) with the same weak dual (b).

Figure 2.13 gives an example of a floorplan and its corresponding coloured weak dual. Here *a* corresponds to an endroom, while *d* and *i* are corner rooms. All of *a*'s incident edges are red, while *d* has edges of both colours, arranged red, red, blue in cyclic order around *d*.

Note that the blue and red edges correspond to the sets *V* and *H* of the floorplan (recall definition 2.17). The rules arise from the properties of rectangular floorplans - namely at any 3-joint, two wall sections run in one direction, and the third in the perpendicular direction, and each room has four walls, two in each direction.

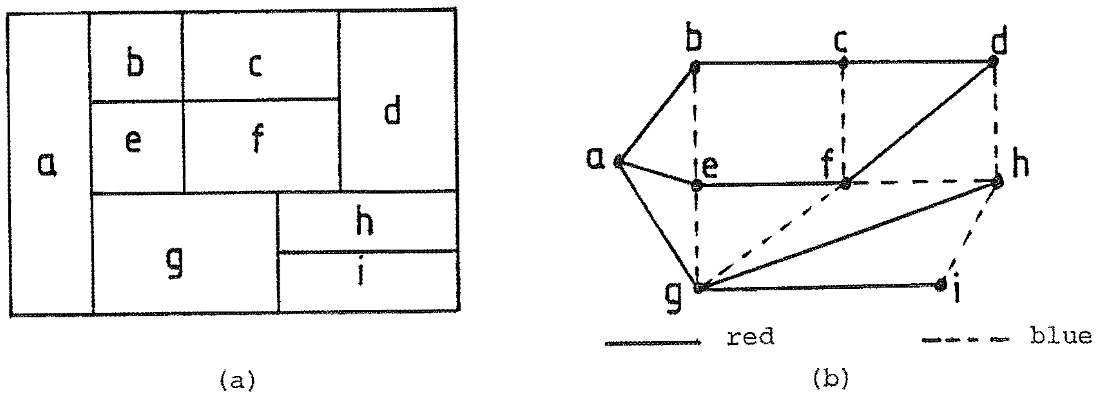


Figure 2.13 Colouring of the weak dual (b) corresponding to floorplan (a).

3. Primary plans

A planar graph to which no edges can be added without making it nonplanar is *maximal planar*. Every face in a maximal planar graph is a triangle. March and Earl (1977) called the set of floorplans having maximal planar adjacency graphs, *primary plans* and showed that all other plans can be obtained from them by processes known as *ornamentation*. Further, they showed primary plans with *n* rooms are related to a smaller number of *fundamental plans*, which in turn have a 1-1 correspondence with trivalent polyhedra with *n*+1 polygonal faces.

4. Graph representations of a dimensioned rectangular floorplan

Both the augmented dual graph and a type of 'electrical' graph can be used to represent the dimensions of a rectangular floorplan. A coloured augmented dual with adjacencies between the exterior regions omitted, can be split into two subgraphs or 'half-graphs' (Mitchell et al, (1976)) corresponding to the two colours.

The vertices in each 'half-graph' represent the rooms and two exterior regions of the plan. Each edge represents a wall section in the plan, and can be weighted by the length of the corresponding wall section. Further each edge can be assigned a direction, so that all edges in each 'half-graph' are directed the same way, say west-east, to form a network. Then the total weight of edges leaving the source, say west, equals the total weight entering the sink, east, and is the overall dimension of the plan from north to south. Further, the total weight of edges entering every other vertex in the 'half-graph' equals the total weight of edges leaving the vertex. This condition corresponds to one of Kirchhoff's two laws for current in an electrical network (see figure 2.14).

Another type of network, introduced by Brooks, Smith, Stone and Tutte (1940) when 'squaring the square', having properties paralleling both of Kirchoff's laws was used by Teague (1970) and March and Steadman (1971). Vertices now represent each maximal continuous straight run of wall running west-east (or north-south), that is, each wall segment. A pair of networks exists for each plan, the sources being the west (or north) sides and the sinks the east (and south) sides of the plan. The edges represent the rooms of the plan. Each vertex is given a value - the horizontal (or vertical) distance of the wall segment it represents from the sink, and each edge the vertical (or horizontal) dimension of the corresponding room.

Either one of the two networks completely represents the dimensioned plan. See figure 2.15 for the west-east network corresponding to the floorplan in figure 2.14.

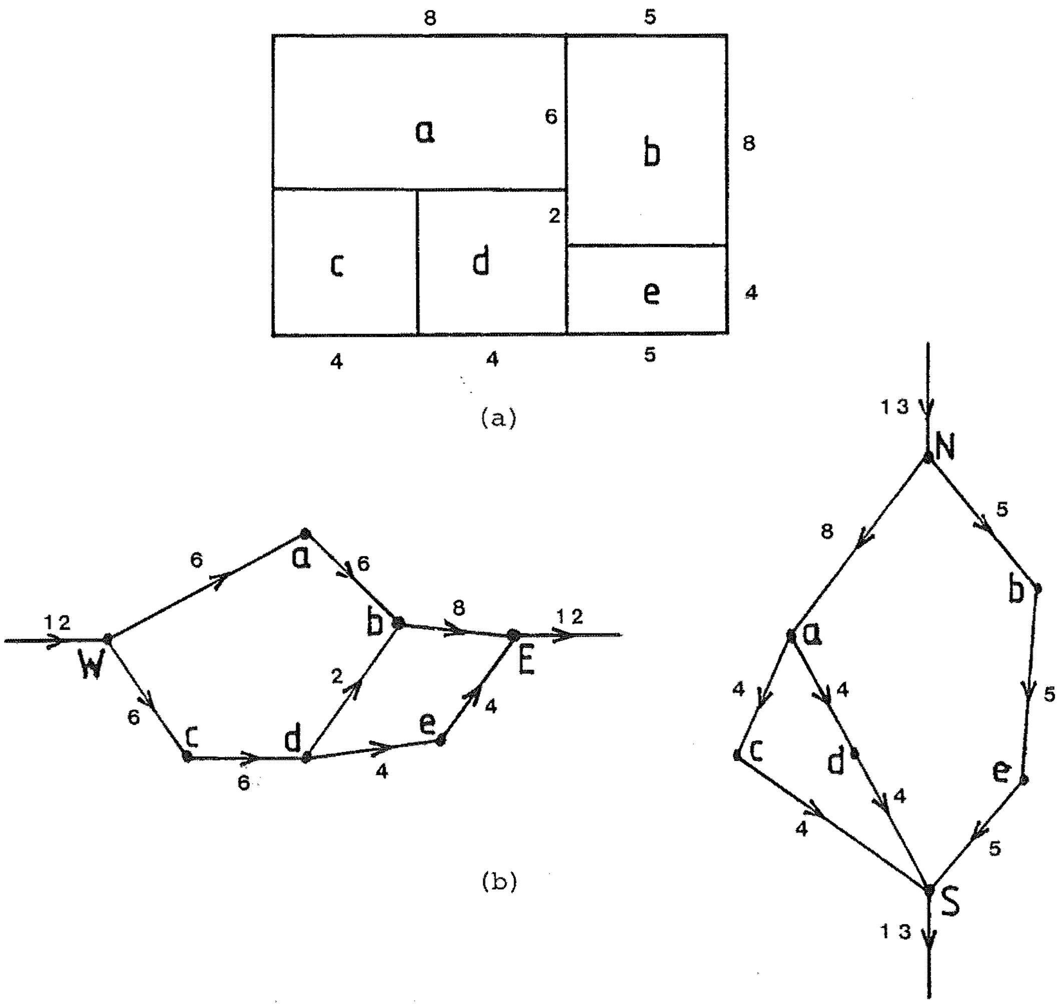


Figure 2.14 The two half-graphs (b) of floorplan (a).

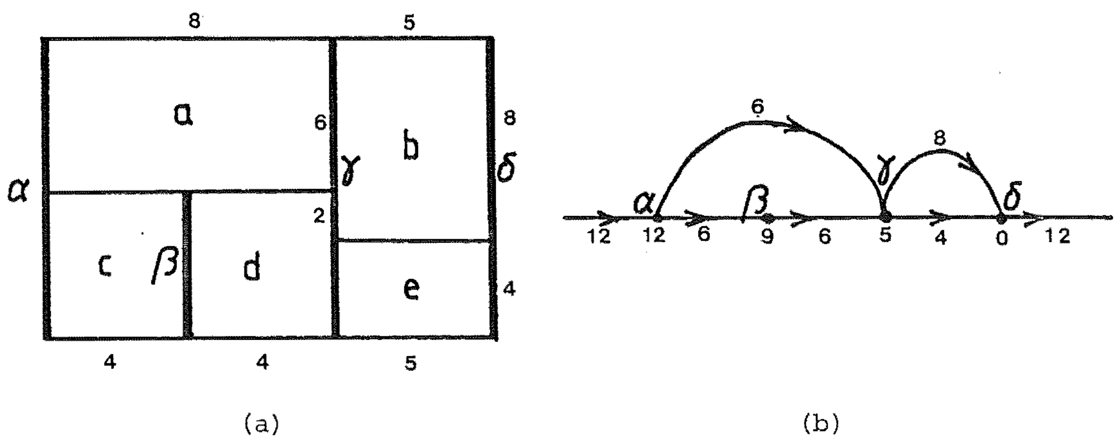


Figure 2.15 The west-east network (b) for floorplan (a).

The two networks contain the same number of edges (rooms) and if the edges entering the sink and leaving the source are omitted, they are duals.

Roth, Hashimshony and Wachman (1982) also used this type of representation but with the weights of the edges representing distances between the walls. Stockmeyer (1983) and Otten (1982a, 1982b) also used similar representations.

E. Designing floorplans

So far we have reviewed ways of counting and representing rectangular plans. We turn now to questions of design in architectural practice and facility layouts - that of finding floorplans that satisfy given constraints.

Usually all the rooms are specified, and some adjacency requirements between rooms, and the exterior. These can be shown in an *adjacency requirement graph*, where vertices represent the rooms, and exterior, and edges the required adjacencies. When a floorplan is drawn, other adjacencies not specified may occur. Thus the adjacency requirement graph will be a spanning subgraph of the adjacency graph of the final plan. There is the possibility that the adjacency requirement graph is nonplanar in which case not every requirement can be satisfied. The problem is to realise the given adjacency graph as a rectangular floorplan. As seen earlier there could be many different floorplans corresponding to the given graph. Also the graph might have more than one embedding in the plane. Further there may be constraints on dimensions and areas of rooms, limiting the choice of floorplans.

Investigations have ranged between heuristic methods intending to generate just one or a few plans in which certain requirements are met, and exhaustive methods producing all possible plans meeting the requirements.

Levin (1964) was the first to apply graph theory to architectural design. Concentrating on access graphs of graphs, he enumerated all outerplanar graphs and their possible labellings up to $n = 4$. He also showed that certain access graphs (namely K_5) cannot be realised in floorplans; that is, no floorplan has K_5 as its access adjacency graph. The second half of his paper dealt with a heuristic method to find an optimal access graph and plan based on cirulation criteria.

Cousin (1970) and Friedman (1975), followed Levin's lead, by looking further at graph-theoretic ideas. The study of floorplans also arose in facility layouts, a branch of operations research. Many authors, for example, Krejčiřík (1969), Seppänen and Moore (1970), Hashimshony, Shaviv and Wachman (1980) were concerned with the problem that the adjacency requirement graph was not planar. They examined tests of planarity, and methods of selecting that minimum 'resolving' set of edges which removed from the adjacency requirement graph changed it into a planar one.

This work was criticized by Steadman (1983) as being unrealistic, for nearly always in architectural practice the adjacency requirement graph is planar.

Grason (1970) was the first to enumerate exhaustively all rectangular floorplans satisfying both adjacency and dimensional requirements. He used the augmented dual divided into two coloured subgraphs as before, with weights on edges representing lengths of walls. Edges were added one at a time to the adjacency requirement graph, checking for 'well formedness' - that is, the subgraph so formed could be coloured to be the augmented dual of a rectangular floorplan, and the weighted edge was consistent with the dimensioning process (similar to the electrical circuits earlier), until all faces were triangles or quadrilaterals. The corresponding dimensioned plan was then derived in all possible ways. Unfortunately the program was very slow, taking 23 minutes, for example, to produce all five solutions to a five-room problem. Grason himself said that for more than eight rooms, a heuristic search technique would need to be developed. Another problem was that the dimensions were given as fixed values.

Gilleard (1978) outlined a related method under development. The question of 'well-formedness' was looked at by Earl and March (1979).

Here they gave the necessary and sufficient conditions for any graph to be the augmented dual or weak dual of a rectangular floorplan without 4-joints. Further they showed that every colouring of such a graph satisfying the colouring rules given earlier, once corners are chosen, produces an oriented floorplan.

A different approach was developed by Mitchell et al (1976) based on the work by Steadman (1973). The first stage found all dimensionless rectangular dissections satisfying the required adjacencies. This was done by searching through a given catalogue of topologically different dissections, with up to eight rectangles (the catalogue mentioned earlier). Dimensions were satisfied by solving a set of simultaneous linear equations which specified lengths, widths or proportions of individual rooms. This was done using either linear programming minimizing or maximizing overall plan length, width, perimeter or proportion. Area requirements require quadratic equations and were solved using nonlinear programming.

Gero (1977) suggested dynamic programming gave better results for the dimensioning stage. However the main disadvantage of this approach was its dependence on the catalogue of rectangular dissections, which as discussed earlier has only been enumerated up to ten rectangles.

Flemming (1978, 1980) developed another two-step method which also satisfied adjacency and dimensional constraints. It was based on his wall representations, and used linear programming for the dimensioning part. His computer program, the DIS program, was capable of exhausting all solutions where constraints were reasonably tight for small number of rooms (Steadman (1983)).

Galle (1981) following the approach of Mitchell et al (1976)developed an exhaustive floorplan generating algorithm for rectangular plans on modular grids which minimized the number of cells in the smallest room. Test results showed realistic problems of up to ten rooms solved in modest computer time.

Baybars (1982) outlined a graph-theoretic technique for the generation of plans without circulation spaces, using an operation called 'wheel expansion' to generate the maximal planar underlying graphs. This operation was outlined in an earlier paper (Baybars and Eastman (1980)) but was later critized by Earl (1981) as being the dual of the face splitting operation of March and Earl (1977) used in ornamentation.

The most recent work similar to Grason is that of Roth et al (1982). Starting from the adjacency requirement graph, non-required adjacencies were added, and the graph split into two subgraphs. These were then converted into networks, vertices representing parallel walls and edges the distance between them, or the dimensions of the rooms. Using the PERT technique for finding the critical path, all edges on the critical path were given their minimal dimension. Combinations of other dimensions were then determined to find a feasible realization. The method appeared successful with as many as twenty rooms, and has been modified to include non-convex rooms and plan boundary.

Korf (1979) criticized the restriction of existing methods to rectangular floorplans and proposed drawing the duals of embedded adjacency graphs as 'bubble diagrams'. However this neglects the fact that any actual floorplan would eventually have to be realised with definite room shapes and dimensions.

In the worked example of Mitchell et al (1976), only six of the 504 dimensionless plans satisfying the adjacency requirements could be dimensioned to suit the area requirements. It seems that, at this point in the investigation, it was unknown whether or not a rectangular floorplan can always be found to satisfy both given adjacency and area requirements. This is one of the questions we are concerned with. As most domestic dwellings require each room to be external, we restrict our attention to floorplans with outerplanar weak duals.

CHAPTER III

MAXIMAL OUTERPLANAR GRAPHS AND TREES

This chapter reviews and introduces the graph theory necessary for the remainder of the thesis. The first section describes outerplanar and maximal outerplanar graphs. In section II, a new index β for trees is introduced, and its properties investigated. The remainder of the chapter concerns the embedding of a tree in a maximal outerplanar graph G, and the relationship between β and the number of vertices of degree 2 in G.

Throughout, the notation and terminology of Harary (1969) is used, unless stated otherwise. In particular, all graphs are finite, loopless, connected and without multiple edges.

I. OUTERPLANAR AND MAXIMAL OUTERPLANAR GRAPHS

Outerplanar and maximal outerplanar graphs have often occurred in the recent literature. This section details their properties, mainly without proof.

Definition 3.1 A planar graph is *outerplanar* if it can be embedded in the plane so that every vertex lies on the exterior face.

Theorem 3.1 [Harary (1969)] A graph G is outerplanar if and only if each of its blocks is outerplanar.

Theorem 3.2 [Harary (1969)] A graph is outerplanar if and only if it contains neither the subgraphs K_4 nor $K_{2,3}$, nor is homeomorphic to these with five or more vertices.

<u>Theorem 3.3</u> [Colbourn and Booth (1981) or Syslo (1979)] Every 2-connected outerplanar graph having at least three vertices possesses a unique hamiltonian cycle.

<u>Corollary 3.4</u> Every vertex of a 2-connected outerplanar graph has degree at least 2.

<u>Corollary 3.5</u> [Mitchell (1979)] Every 2-connected outerplanar graph with n vertices is isomorphic to an n-gon divided into polygons by chords.

<u>Definition 3.2</u> An outerplanar graph G with n vertices is *maximal outerplanar* if no edge can be added to G without losing outerplanarity.

<u>Theorem 3.6</u> [Harary (1969)] Every interior face of a maximal outerplanar graph is a triangle.

<u>Theorem 3.7</u> [Harary (1969)] Every maximal outerplanar graph with n vertices is a triangulation of some polygon P_n with n points.

<u>Theorem 3.8</u> [Harary (1969)] Let G be a maximal outerplanar graph with $n \geq 3$ vertices.

Then G has

(i) $2n-3$ edges

(ii) $n-2$ interior faces

(iii) at least two vertices of degree 2

(iv) at least three vertices of degree not exceeding 3.

<u>Theorem 3.9</u> [Mitchell (1979)] Let G be a maximal outerplanar graph with $n \geq 3$ vertices. Then G does not have a vertex u of degree 2, whose two neighbours v and w are not adjacent.

Corollary 3.10 A maximal outerplanar graph G with more than three vertices does not have two vertices of degree 2 adjacent.

Proof: (By contradiction). Assume u and v are adjacent vertices of degree 2 in G. Let their other neighbouring vertices be w and x respectively, with $w \neq x$. Then from theorem 3.9 v and w are adjacent so that v has degree at least 3, contradicting the original assumption. #

More important is the following well known result concerning the recursive construction of any maximal outerplanar graph. (See Proskurowski (1979), for example.)

Theorem 3.11 A graph is maximal outerplanar if and only if it can be constructed from a triangle (K_3) by a finite number of applications of the following procedure: to the graph already constructed add a new vertex u in the exterior face and join it to two vertices v and w adjacent in the exterior face.

Figure 3.1 shows a maximal outerplanar graph constructed in this manner from the initial triangle labelled 1,2,3. The i^{th} vertex for $i \geq 3$ is joined to two vertices having labels less than i.

Definition 3.3 From a graph G, the subgraph G − u is formed by deleting vertex u and all edges incident with u.

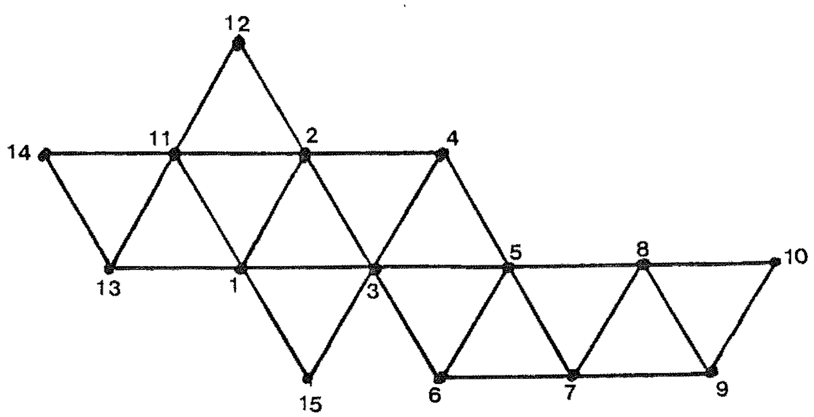


Figure 3.1 A maximal outerplanar graph

Theorem 3.12 [Mitchell (1979)] If G is a maximal outerplanar graph with $n > 3$ vertices, then for any vertex u with degree 2 in G, G - u is maximal outerplanar.

Lemma 3.13 Let G be a maximal outerplanar graph with hamiltonian circuit C. For every two distinct vertices r and s in G, there exist two distinct paths consisting entirely of edges in C from r to s; one going clockwise from r to s around the circuit, the other anticlockwise.

Theorem 3.14 Let G be a maximal outerplanar graph with hamiltonian circuit C. Let r,s,t,u be distinct vertices in G with r joined to s by an edge not in C, and t joined to u by an edge not in C. Then both t and u are in the same path consisting entirely of edges in C from r to s.

Proof: If t and u are in different paths consisting entirely of edges in C, we have the situation shown in figure 3.2. The edges $\{r,s\}$ and $\{t,u\}$ intersect at a point within the hamiltonian circuit which is a contradiction, since G is maximal outerplanar. #

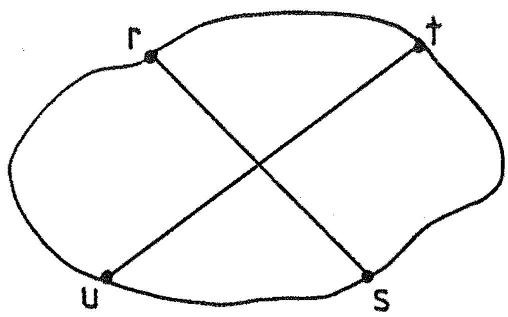


Figure 3.2 The situation for theorem 3.14.

Theorem 3.15 Let G be a maximal outerplanar graph with hamiltonian circuit C, and let r and s be two distinct vertices joined by an edge not in C. Further let $w_1, w_2, \ldots w_i$ be the vertices in order as they occur in one of the paths consisting of edges only in C from r to s. Then the subgraph H of G induced by the vertices $w_1, w_2, \ldots w_i, r, s$ is maximal outerplanar and at least one w_j has degree 2 in G.

Proof: The subgraph H has a hamiltonian circuit passing through the vertices r, $w_1, w_2, \ldots w_i$, s in order, and hence is outerplanar. No w_j is adjacent in G to a vertex u that is in G but not in H, for then they would be joined by an edge not in C and would be in different paths consisting entirely of edges in C from r and s, contradicting theorem 3.14. Each w_j is therefore adjacent to the same vertices in H as in G. As G is maximal outerplanar, by theorem 3.6, each interior face in G is a triangle. Thus each interior face in H is also a triangle, and H is maximal outerplanar.

Vertices r and s do not have the same degree in H as in G, but since they are adjacent in H, by corollary 3.10, at most one of them has degree 2 in H. By theorem 3.8, H must have at least one vertex other than r or s with degree 2. Thus at least one w_j has degree 2 in H, and so also has degree 2 in G. #

A. Enumerating maximal outerplanar graphs

Beineke and Pippert (1972) showed that the number of nonisomorphic maximal outerplanar graphs with n vertices, $|G_n|$, where $n > 3$ is given by

$$|G_n| = \frac{1}{2n}\, t(n-2) + \frac{1}{2}\, t\left(\frac{n-3}{2}\right) + \frac{3}{4}\, t\left(\frac{n-2}{2}\right) + \frac{1}{3}\left(\frac{n-3}{3}\right)$$

$$\text{where} \quad t(x) = \begin{cases} \dfrac{(2x)!}{x!\,(x+1)!} & \text{if } x \text{ integer} \\ 0 & \text{otherwise.} \end{cases}$$

We end this section by giving in table 3.1 the number of nonisomorphic maximal outerplanar graphs, where n < 13, divided into groups according to the number of vertices of degree 2. It can be seen that as n increases, the number of such graphs grows at an accelerating rate.

Table 3.1. The number of nonisomorphic maximal outerplanar graphs $|G_n|$, divided into groups according to the number of vertices with degree 2.

Number of vertices	Number of vertices of degree 2				Total number of graphs
n	2	3	4	>4	$\|G_n\|$
3		1			1
4	1				1
5	1				1
6	2	1			3
7	3	1			4
8	6	5	1		12
9	10	14	3		27
10	20	42	19	1	82
11	36	112	73	7	228
12	72	304	295	62	733

II BRANCHING INDEX OF TREES

Definition 3.4 A tree with n vertices in which every vertex has degree 1 or 2 is called a *path graph* P_n. An isolated point is considered as P_1. A path graph P_n with one vertex being a root, is called a *rooted path graph*. A tree is *branching* if it is not a path graph.

Definition 3.5 Given a tree T with root u, delete the root. What is left is a collection of k subtrees, where k is the degree of u. Each subtree is taken as rooted at the vertex that was initially adjacent to u. The number of these rooted subtrees which are branching is called the *branching index* of u, and denoted by $\beta(u)$ or β. The branching index for each vertex of an unrooted tree is obtained by treating each vertex in turn as the root of the tree.

<u>Lemma 3.16</u> Let u be a vertex in tree T with $\beta(u) \geq 2$. If $w \neq u$ is another vertex in T, then the subtree of $T-w$ which contains u is branching.

<u>Proof:</u> Since $\beta(u) \geq 2$ and w is one of the subtrees of $T-u$, there are at least $\beta(u)-1$ subtrees of $T-u$ which are branching and do not contain w. These are included in the subtree of $T-w$ containing u. The result follows. #

<u>Lemma 3.17</u> Let u be a vertex in tree T with one of the subtrees in $T-u$ being a path graph. Let Z be this subtree. Then every vertex w in Z has branching index at most 1 in T.

<u>Proof:</u> For each w in Z, all the subtrees of $T-w$ not containing u are subtrees of Z and hence are path graphs. Thus only the subtree of $T-w$ containing u can be branching. The result follows. #

<u>Theorem 3.18</u> In any tree T the vertices with $\beta \geq 2$ induce a subtree, or there are no such vertices.

<u>Proof:</u> Let B be the set of vertices with $\beta \geq 2$. If B is not empty, assume the subgraph of T induced by the vertices in B is not connected. Then there exist two vertices u and v in B which are not adjacent in T, and another vertex w not in B, which is on the path joining u to v in T. Thus u and v are in different subtrees of $T-w$. Since $\beta(u) \geq 2$, by lemma 3.16 the subtree of $T-w$ containing u is branching. Similarly the subtree of $T-w$ containing v is branching. Hence $T-w$ contains at least two branching subtrees, implying $w \in B$ which contradicts the original assumption. Thus the subgraph of T induced by B is connected and is a tree. #

<u>Theorem 3.19</u> If in any tree T, all vertices have $\beta \leq 2$, then either the subtree induced by vertices having $\beta = 2$ is a path graph, or there are no such vertices.

Proof: If no vertices have $\beta = 2$, the result follows. However if there are vertices in T having $\beta = 2$, assume the subtree S induced by them is not a path graph. Then there exist distinct vertices w, x, y, z in S, where w is on the path in S between x and y, but z is not, and each of w,x,y,z has $\beta = 2$.

The situation is as shown in figure 3.3.

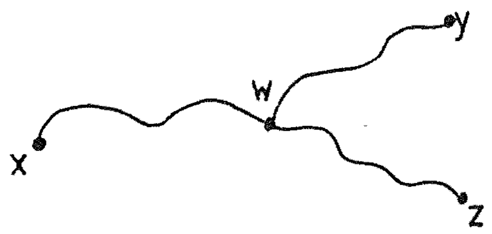


Figure 3.3 The situation in theorem 3.19.

Since $\beta(y) = 2$, by lemma 3.16, the subtree of $T-w$ containing y is branching. Similarly, the subtree of $T-w$ containing z, and that containing x are branching. Thus $\beta(w) \geq 3$ which contradicts the original assumption. #

Remarks: All vertices in a tree can have branching index of 0. This only occurs if the tree is itself a path graph or is that shown in figure 3.4.

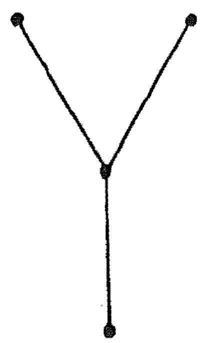


Figure 3.4 A tree in which each vertex has $\beta = 0$.

Trees with all vertices having $\beta = 0$ or 1 only also exist. Some examples of these are shown in figure 3.5. Each vertex has its branching index written alongside.

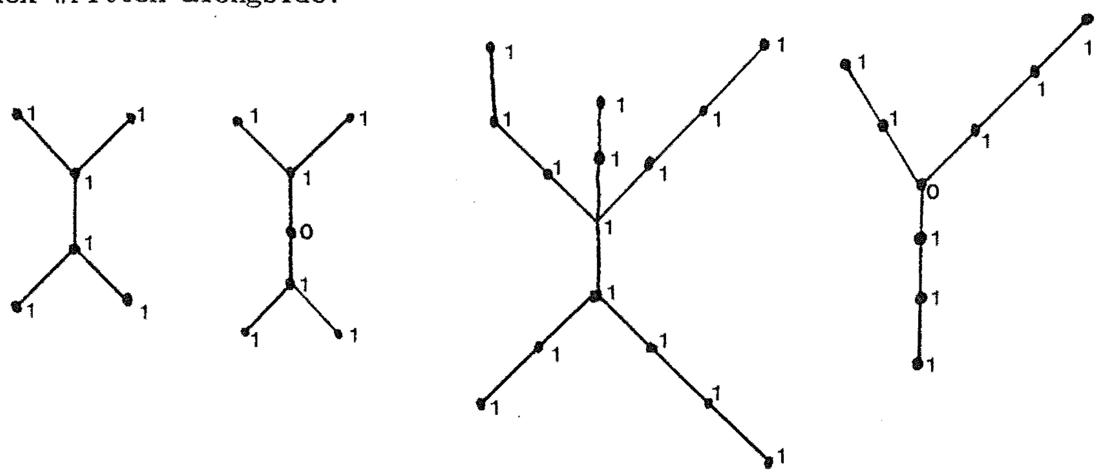


Figure 3.5 Trees in which each vertex has $\beta = 0$ or 1.

Lemma 3.20 A tree with vertex u having $\beta(u) = 0$ cannot have any vertex v with $\beta(v) \geq 2$.

Proof: From lemma 3.16. #

Definition 3.6 The *linking subtree* of a tree T induced by vertices in the set $V = \{a, b, c, \ldots, r\}$ is the subtree of T induced by V and all vertices u in T but not in V lying on a path in T between any two distinct elements of V. It is thus the minimum subtree of T containing set V.

Definition 3.7 For any vertex x in a tree T, $\deg_T(x)$ denotes the degree of x in T.

Theorem 3.21 Let T be a branching tree, and let D be the linking subtree of T induced by vertices x with $\deg_T(x) \geq 3$. Every vertex in T has branching index of 0 or 1 if and only if

either (i) exactly one vertex x in T has $\deg_T(x) \geq 3$,

or (ii) the diameter of D is d, where $1 \leq d \leq 3$ and at most $3-d$ terminal vertices of D correspond to vertices having degree at least 3 in T.

Proof: We first prove the sufficiency.

(i) If exactly one vertex x has $\deg_T(x) \geq 3$, each of the subtrees in $T-x$ is a path graph. Thus $\beta(x) = 0$, and every other vertex v in T has $\beta(v) \leq 1$ by lemma 3.17.

(ii) Assume at least two vertices in T have degree at least 3 and let D be the linking subtree induced by them. Let the diameter of D be d.

If $d = 1$, then D consists of two vertices u, v joined by an edge. Every other vertex in T has degree 1 or 2. Thus each subtree of $T-u$ except that containing v is a path graph. Let Z be one of these subtrees. By lemma 3.17, each vertex w in Z has $\beta(w) \leq 1$ in T. If the subtree of $T-u$ containing v is branching, $\beta(u) = 1$. Otherwise $\beta(u) = 0$. Similarly $\beta(v) \leq 1$, and any vertex y in a subtree of $T-v$ not containing u, has $\beta(y) \leq 1$. So each vertex x in T has $\beta(x) = 0$ or 1.

If $d=2$ and at most one terminal vertex u in D has $\deg_T(u) > 3$, then D is as shown in figure 3.6 where $\ell \geq 1$. From the definition of D, it follows $\deg_T(w_j) = 3$ and the subtree of $T-v$ containing w_j is a path graph, for every j from 1 to ℓ. Thus as above, each vertex x in T has $\beta(x) = 0$ or 1.

If $d = 3$ and no terminal vertex y in D has $\deg_T(y) > 3$, then D is as shown in figure 3.7.

Here $k \geq 1$ and $\ell \geq 1$, and $\deg_T(z_i) = 3$ for each i from 1 to k. Also $\deg_T(w_j) = 3$ for each j from 1 to ℓ. All vertices in T but not in D

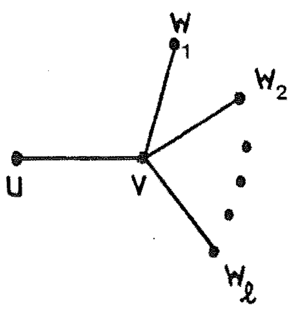


Figure 3.6 A tree with $d = 2$.

have degree 1 or 2. Thus each subtree of $T-u$ not containing v is a path graph. Similarly each subtree of $T-v$ not containing u is a path graph. Thus from lemma 3.17 and above, each vertex x in T has $\beta(x) = 0$ or 1.

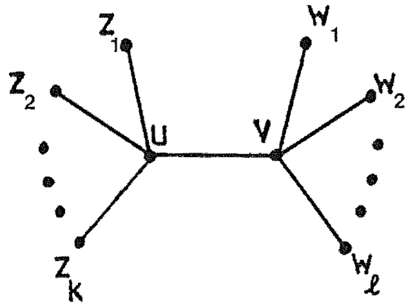


Figure 3.7 A tree with $d = 3$.

To prove the necessity, let T be a branching tree in which each vertex has $\beta(x) = 0$ or 1. Then exactly one vertex y in T may have $\deg_T(y) \geq 3$. Otherwise, assume D has diameter $d > 3$. Then there exists two terminal vertices in D, u and v, and a path $\langle u, w_1, w_2, \ldots, w_r, v \rangle$ from u to v in D, with $r \geq 3$. The subtree of $T-w_2$ containing w_1, also contains u, and since $\deg_T(u) \geq 3$ it is branching. Similarly the subtree of $T-w_2$ containing w_3 also contains v and is branching. Thus $\beta(w_2) \geq 2$ which is a contradiction. So $d \leq 3$.

If $d = 3$, D is of the form shown in figure 3.7. Suppose some terminal vertex z_1 has $\deg_T(z_1) > 3$. As $\deg_T(w_1) \geq 3$, it follows that both the subtree of $T-u$ containing z_1 and the one containing w_1, are branching. Thus $\beta(u) \geq 2$, a contradiction.

If $d = 2$ and at least two terminal vertices, u and w_1 say, have degree greater than 3 in T, then by a similar argument considering figure 3.6, $\beta(v) \geq 2$, which is a contradiction.

If $d = 1$, so that D consists of two adjacent vertices u, and v, both $\deg_T(u) \geq 3$ and $\deg_T(v) \geq 3$.

The result of the theorem follows. #

<u>Lemma 3.22</u> Every tree with n vertices, where $n > 2$, can be embedded as a spanning tree of a maximal outerplanar graph with n vertices.

<u>Proof:</u> Given a tree T, select any vertex to be the root. Perform a depth-first search of the vertices starting from the root (Aho, Hopcroft and Ullman (1974)). Number the vertices 1, ...n in increasing order as they are visited. Redraw the tree with the root at the top, and let the neighbours of each vertex x with labels greater than x be drawn so that they occur in increasing order from left to right across the page. Add edges to T, where necessary, joining each vertex i to vertex i+1 for i ranging from 1 to n-1, and joining vertex n to the root. An outerplanar graph with an hamiltonian circuit passing through the vertices in numerical order results. Adding edges to triangulate each interior face of the graph results in a maximal outerplanar graph G. #

Figure 3.8 shows a tree embedded in a maximal outerplanar graph in the manner outlined by lemma 3.22.

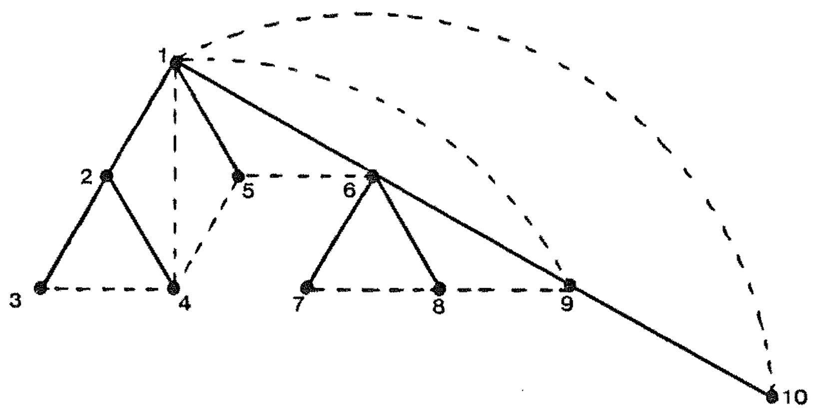


Figure 3.8 A tree T embedded in a maximal outerplanar graph G. T is shown by solid lines. Additional edges to T to form G are shown dotted.

Theorem 3.23 [Nebesky (1976)] Let T be a spanning tree of a maximal outerplanar graph G with hamiltonian cycle C. If vertices r,s,t,u are such that t and u belong to distinct components of the graph $C-r-s$, then the path in T from r to s, and the path in T from t to u have at least one vertex in common.

Theorem 3.24 Any maximal outerplanar graph G with n vertices has a minimum of 2 and a maximum of $\frac{n}{2}$ if n is even, or $\frac{n}{2}-1$ if n is odd, vertices of degree 2.

Proof: From theorem 3.8 and corollary 3.10. #

Theorem 3.25 Let T be a spanning tree of a maximal outerplanar graph G with $n > 3$ vertices and hamiltonian cycle C. Let D be the linking subtree of T induced by the vertices having degree 2 in G. This tree consists of m vertices $u_1,u_2,\ldots,u_m$ of T. Let the remaining vertices of T be labelled $v_1,v_2,\ldots,v_{n-m}$ in order as they occur clockwise around C.

Then (i) if v_j is adjacent to v_k in T, $\{v_j,v_k\}$ is an edge in C,

(ii) if v_j is adjacent to some u_i in T, the subtree of $T-u_i$ containing v_j is a path graph,

and (iii) $\beta(v_j) \leq 1$ in T,

for each j between 1 and $n-m$.

Proof: (i) Assume $\{v_j,v_k\}$ is an edge not in C. Then by theorem 3.15, there is a vertex u_k having degree 2 in G, which is in the path consisting entirely of edges in C from v_j to v_k and going clockwise around the circuit. Similarly, there is a vertex u_y with degree 2 in G in the path consisting only of edges in C from v_j to v_k and going anticlockwise around the circuit. Thus (when G is embedded in the plane), we have the situation shown in figure 3.9.

Vertices u_x and u_y belong to different components of $C-v_j-v_k$, and hence from theorem 3.23, the path in T from v_j to v_k, and the path in T

from u_x to u_y have at least one vertex in common. Since the path from υ_j to υ_k in T is the edge $\{\upsilon_j, \upsilon_k\}$, the vertex common to both paths is either υ_j or υ_k. But, from the definition of D, each vertex in the path from u_x to u_y is labelled some u_i. Thus we have a contradiction.

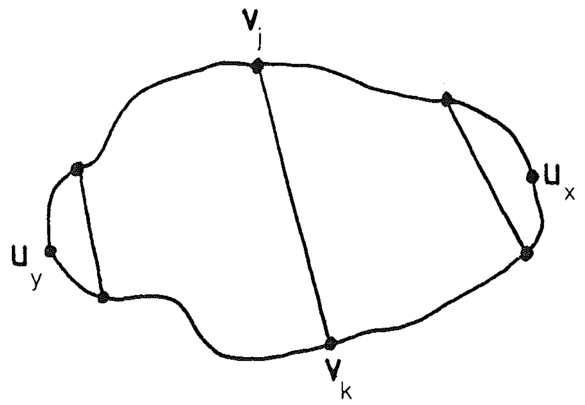


Figure 3.9 The situation in theorem 3.25.

(ii) In T, υ_j can be adjacent to at most one of $\{u_1, u_2, \ldots u_m\}$ or else there is a circuit. Because of the way in which the vertices were labelled and (i) above, only $\upsilon_{j-1(mod\ m)}$, or $\upsilon_{j+1(mod\ m)}$, or both can be neighbours of υ_j in T provided $\{\upsilon_j, \upsilon_{j-1(mod\ m)}\}$ or $\{\upsilon_j, \upsilon_{j+1(mod\ m)}\}$ respectively are edges in C.

The possibilities for υ_j in T, taking subscripts modulo m, therefore are:-

(1) degree 1, adjacent only to some u_i;

(2) degree 2, adjacent to u_i and one of υ_{j-1} or υ_{j+1};

(3) degree 3, adjacent to u_i and both υ_{j-1} and υ_{j+1};

(4) degree 1, not adjacent to any u_i, adjacent to υ_{j-1} (or V_{j+1});

(5) degree 2, not adjacent to u_i, adjacent to both υ_{j-1} and V_{j+1}.

If υ_j is adjacent to u_i, the subtree (which may be quite long) of $T-u_i$ containing υ_j cannot contain any other vertex u_ℓ, or vertex υ_s adjacent in T to any u_ℓ because either would imply a circuit. The subtree of T-u_i containing v_j must therefore be a path graph.

(iii) If v_j is not adjacent in T to some u_i, then since T is connected, there is a path in T from v_j to some v_k with v_k adjacent in T to some u_r. The subtree of T-u_r containing v_k also contains v_j and by (ii) above is a path graph. Hence by lemma 3.17, $\beta(v_j) \leq 1$. Also by lemma 3.17 and (ii) above, if v_j is adjacent to some u_i, $\beta(v_j) \leq 1$.

Thus all three cases of the theorem hold. #

A given tree T with n vertices can be embedded as a spanning tree of two maximal outerplanar graphs G_1 and G_2 both with n vertices but with differing numbers of vertices having degree 2. Figure 3.10 shows an example of this.

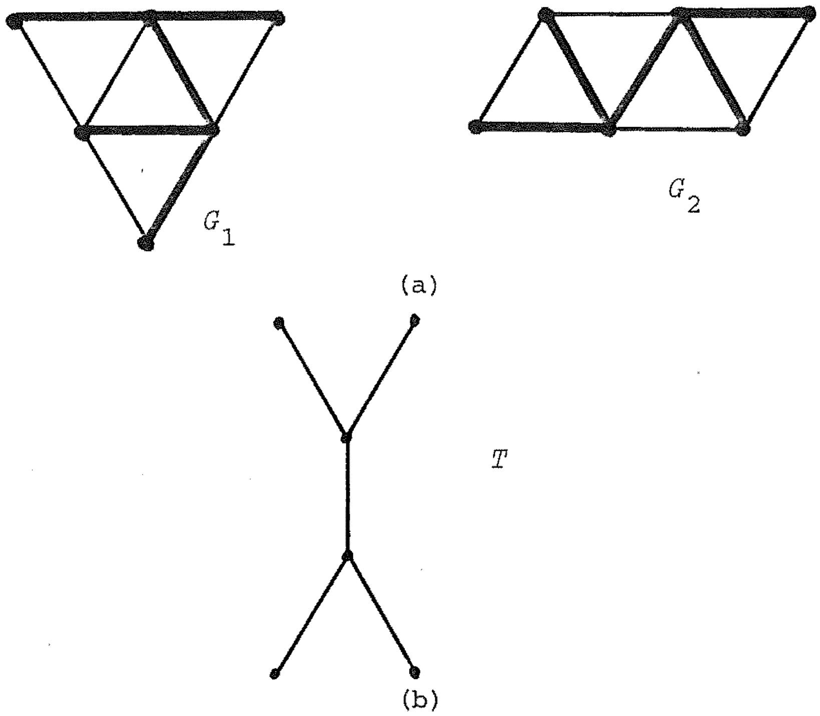


Figure 3.10 Two maximal outerplanar graphs with different numbers of vertices of degree 2(a) having the same spanning tree (b).

We now establish the exact minimum number of vertices of degree 2 for a maximal outerplanar graph in which a given tree may be embedded.

First the necessary and sufficient conditions for there to exist an maximal outerplanar graph with two vertices of degree 2, within which the given tree can be embedded, are established. An extension to the general case then follows. The results of the previous section are required.

A. Two vertices of degree 2

Theorem 3.26 A tree T with n vertices, where $n > 3$, is isomorphic to a spanning tree of some maximal outerplanar graph with exactly two vertices of degree 2 if and only if the branching index of each vertex in T is at most 2.

Proof: To prove the necessity, consider a maximal outerplanar graph G with $n > 3$ vertices having two vertices of degree 2. Let T be a spanning tree of G. Then the linking subtree D of T induced by the vertices having degree 2 in G is a path $\langle u_1, u_2, \ldots, u_m \rangle$ of length m connecting the two vertices u_1 and u_m of degree 2 in G. Let the remaining vertices of T be labelled $v_1, v_2, \ldots, v_{n-m}$ in order as they occur clockwise around the hamiltonian cycle of G.

Consider the branching index of each vertex w in T. If w is one of $u_1, u_2, \ldots, u_m$, say u_i, then u_i is adjacent to u_{i-1} provided $i \neq 1$, u_{i+1} provided $i \neq m$, and possibly some of $\{v_1, v_2, \ldots, v_{n-m}\}$. The rooted subtrees of $T-u_i$ rooted at the vertices adjacent to u_i consist of one rooted at u_{i-1} if $i \neq 1$, another rooted at u_{i+1} if $i \neq m$, and possibly some rooted path graphs by part (ii) of theorem 3.25. Hence $\beta(u_i) \leq 2$.

Also by theorem 3.25, $\beta(v_j) \leq 1$ for each j between 1 and $n-m$. Thus each vertex in the tree has branching index at most 2.

We now prove the sufficiency. Let T be a tree with n vertices, where $n > 3$, and with the branching index of each vertex being at most 2. Then from theorem 3.19, either the subgraph induced by the vertices having $\beta = 2$ is a path graph, or there are no such vertices.

(i) Consider first the case when there are m vertices in T, where $m \geq 1$, each having $\beta = 2$. The subgraph P induced by these vertices is a path graph. If $m = 1$, let the only vertex in T with $\beta = 2$ be labelled u_2. Otherwise label the vertices of T with $\beta = 2$ as $u_2, u_3, \ldots, u_{m+1}$ in order as they occur in the path P.

If $m = 1$, two of the subgraphs of $T - u_2$ are branching. Both these contain a vertex adjacent to u_2 in T. Label these vertices as u_1 and u_3 respectively.

However if $m > 1$, then since $\beta(u_2) = 2$, u_2 must have an adjacent vertex in T, other than u_3, for which the subtree of $T-u_2$ containing this vertex is branching. Label this vertex u_1. Label the neighbour of u_{m+1}, other than u_m, for which a similar result holds, as u_{m+2}.

Then the subtrees of $T-u_k$, for any k from 1 to $m+2$, rooted at unlabelled neighbours of u_k are all rooted path graphs. They are all one of the three types shown in figure 3.11.

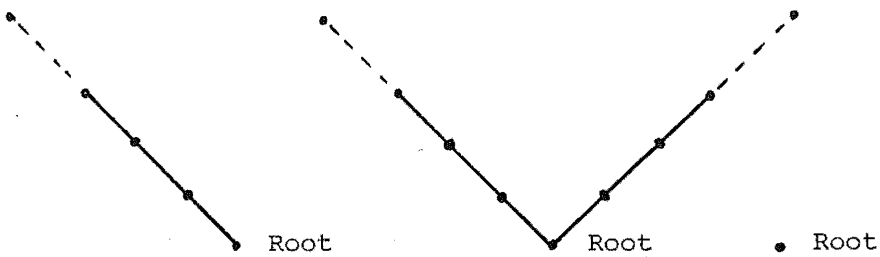


Figure 3.11 The three types of rooted path graphs.

<u>Algorithm 3.1</u> Any remaining unlabelled vertices of the spanning tree T are labelled $v_1, v_2, \ldots, v_{n-m-2}$ as follows:-

<u>Step 1</u>: Consider a rooted path graph of $T-u_k$, all of whose vertices are unlabelled, of type (i) in figure 3.11. If none exist, go to <u>Step 3</u>.

Label the terminal vertex other than the root in this subtree. Continue labelling the unlabelled neighbour of the previously labelled vertex until the root is reached. Label the root.

Step 2: Repeat Step 1.

Step 3: Consider a rooted path graph of $T-u_k$, all of whose vertices are unlabelled, of type (ii) in figure 3.11. If none exist, go to Step 5.

Label one of the terminal vertices of this tree. Continue labelling the unlabelled neighbour of the previously labelled vertex until the other terminal vertex is labelled.

Step 4: Repeat Step 3.

Step 5: Consider an unlabelled rooted path graph of $T-u_k$ of type (iii) in figure 3.11. If none exist, the algorithm ends.

Label the root.

Step 6: Return to Step 5.

For each u_k, the vertices in the set $\{v_p, v_{p+1}, \ldots, v_{p+q}\}$ are labelled. See figure 3.12, which shows a tree labelled this way, with $\beta(u_2) = \beta(u_3) = 2$.

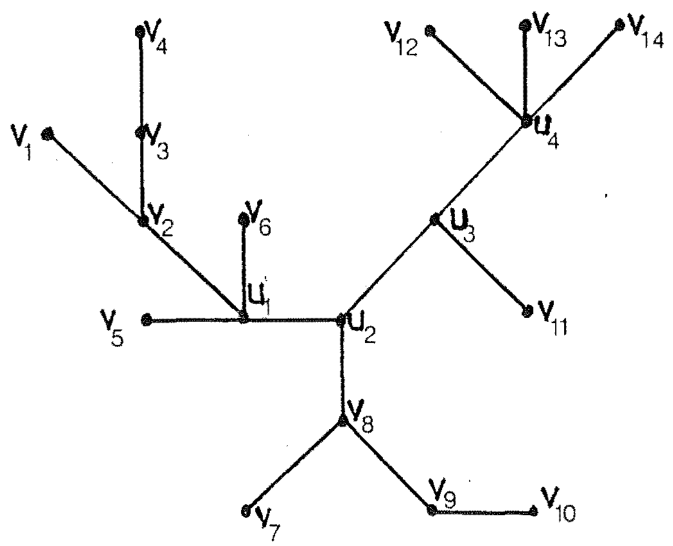


Figure 3.12 A tree labelled according to the sufficiency part of theorem 3.27.

The tree T is now redrawn with its vertices in two lines across the page; $\upsilon_1, \upsilon_2, \ldots, \upsilon_{n-m-2}$ in order in the upper line and $u_1, u_2, \ldots, u_{m+2}$ in order in the lower. If the tree is drawn with the vertices evenly spaced along the lines, and the edges represented by straight lines, no lines cross improperly since the υ_i vertices belonging to any u_k occur consecutively and the groups occur in the same order as the u_k.

Add edges to tree T in the following manner:-

(i) join u_1 to υ_1, if not already adjacent;

(ii) join u_{m+2} to υ_{n-m-2}, if not already adjacent; and

(iii) join each υ_j to υ_{j+1}, if not already adjacent, for each j from 1 to $n-m-3$.

(iv) In algorithm 3.1 a set of vertices $\{\upsilon_p, \upsilon_{p+1}, \ldots, \upsilon_{p+q}\}$ was labelled for each u_k. Join each of these vertices to u_k, and join υ_{p+q} to u_{k+1} if $k \neq m+2$, if not already adjacent. Repeat for each u_k.

A maximal outerplanar graph G with hamiltonian circuit passing through $\upsilon_1, \upsilon_2, \ldots, \upsilon_{n-m-2}, u_{m+2}, u_{m+1}, \ldots, u_2, u_1$, in sequence and having two vertices of degree 2, namely υ_1 and υ_{n-m-2}, is formed. Figure 3.13 illustrates how edges were added to the tree of figure 3.12 forming a maximal outerplanar graph with two vertices of degree 2, namely υ_1 and υ_{14}. Here $\upsilon_1, \upsilon_2, \ldots, \upsilon_5, \upsilon_6$ were formed from u_1 in algorithm 3.1 and so are joined to u_1, while υ_6 is also joined to u_2.

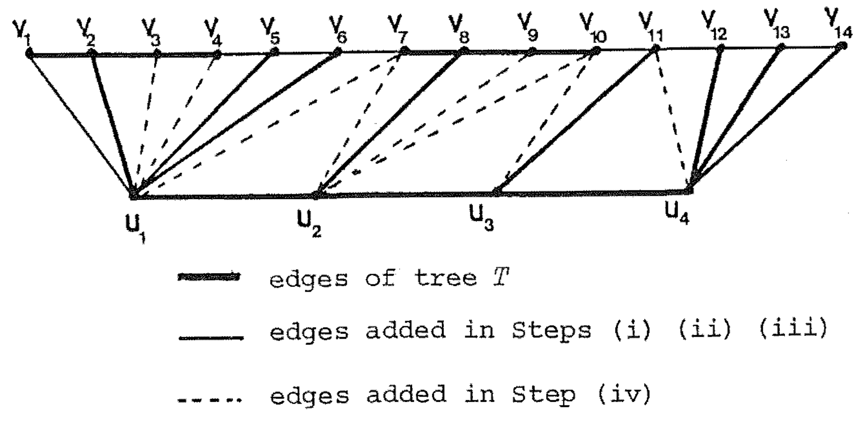


Figure 3.13 Addition of edges to tree T in figure 3.12, after being redrawn, to form a maximal outerplanar graph with two vertices of degree 2.

(ii) Consider now the case where no vertex in T has branching index of 2. Then from theorem 3.21 earlier, either T is a path graph or the linking subtree D of T induced by vertices having degree at least 3 is of type (a), (b) or (c) in figure 3.14.

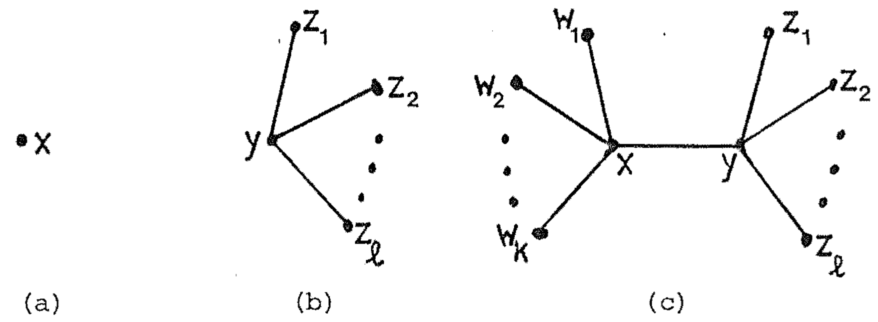


Figure 3.14 The three types of linking subtree D of T, induced by vertices with degree at least 3 when every vertex x in T has $\beta(x) \leqslant 1$.

If T is a path graph label one of its vertices as u_1 and set $m = -1$. Otherwise if the subtree D of T induced by vertices with degree at least 3 is type (a) or (b) in figure 3.14, label the vertex of T corresponding to x or y as u_1. Should D be of type (c), label the vertices of T corresponding to x and y as u_1 and u_2 respectively. In each case set $m = 0$.

Use algorithm 3.1 to label the remaining vertices in the tree, and add edges as before to form a maximal outerplanar graph G with exactly two vertices of degree 2.

Both parts of the theorem are now proved. #

Remark: Consider the tree in figure 3.15, in which $\beta(u) = 3$. It can be shown that no tree having at most twelve vertices has a vertex u with $\beta(u) \geq 3$. Hence from theorem 3.26, this is the tree with the smallest number of vertices which is not the spanning tree of any maximal outerplanar graph with exactly two vertice of degree 2.

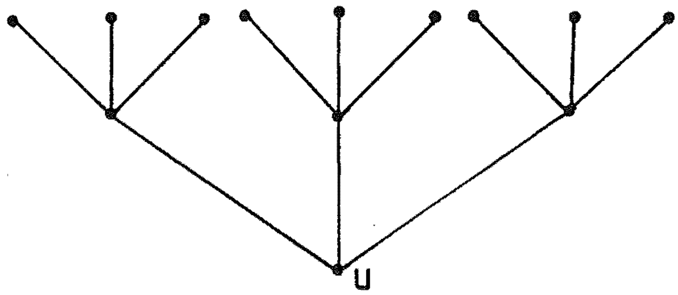


Figure 3.15 A tree with the least number of vertices having one vertex u with $\beta(u) = 3$.

B. General case

Definition 3.8 For any tree T, the sum $\sum_{u \in T : \beta(u) > 2} (\beta(u)-2)$ is called the *embedding index* of T and is denoted by $\epsilon(T)$ or ϵ. If no vertex u has $\beta(u)>2$, then $\epsilon(T) = 0$.

Theorem 3.27 Let G be a maximal outerplanar graph with $n > 3$ vertices, of which $p \geq 2$ have degree 2 and let T be a spanning tree of G. Then $\epsilon(T) \leq p-2$.

Proof: If $p = 2$, then from theorem 3.26, no vertex u has $\beta(u)>2$ and $\epsilon(T) = 0 = p-2$.

If $p \geq 3$, let D be the linking subtree of T induced by the p vertices of degree 2 in G. This tree consists of m vertices $u_1, u_2, \ldots, u_m$ from T. Let the remaining vertices of T be labelled $v_1, v_2, \ldots, v_{n-m}$ in order as they occur clockwise around the hamiltonian circuit of G.

Then from theorem 3.25, every v_j, where j lies between 1 and $n-m$, has $\beta(v_j) \leq 1$ in T.

A vertex u in T with $\beta(u) \geq 2$ in T corresponds to some u_k in subtree D. Further $\beta(u) \leq deg_D(u)$, the degree of u in D, for every u in D.

Tree D contains m vertices - r with degree 1, where $r \leq p$, (vertices corresponding to the degree 2 vertices of G), s with degree 2 and t with degree greater than 2. Note that not every vertex of degree 2 in G must correspond to a terminal vertex in D. In figure 3.16, for example, only five of the six vertices with degree 2 in G have degree 1 in D, as $deg_D(u_3) = 2$.

So $$\sum_{k=1}^{m} deg_D(u_k) = 2(m-1)$$

$$= 2(r+s+t-1) \text{ counting vertices.}$$

Also $$\sum_{k=1}^{m} deg_D(u_k) = r + 2s + \sum_{u\epsilon D:deg_D(u)>2} deg_D(u).$$

So $$\sum_{u\epsilon D:deg_D(u)>2} deg_D(u) = r + 2t - 2$$

$$\Rightarrow \sum_{u\epsilon D:deg_D(u)>2} deg_D(u) - 2t = r-2$$

$$\Rightarrow \sum_{u\epsilon D:deg_D(u)>2} (deg_D(u) - 2) = r-2$$

$$\leq p-2$$

So $$\epsilon(T) = \sum_{u\epsilon T:\beta(u)>2} (\beta(u) - 2)$$

$$\leq \sum_{u\epsilon D:deg_D(u)>2} (deg_D(u) - 2)$$

$$\leq p-2.$$ #

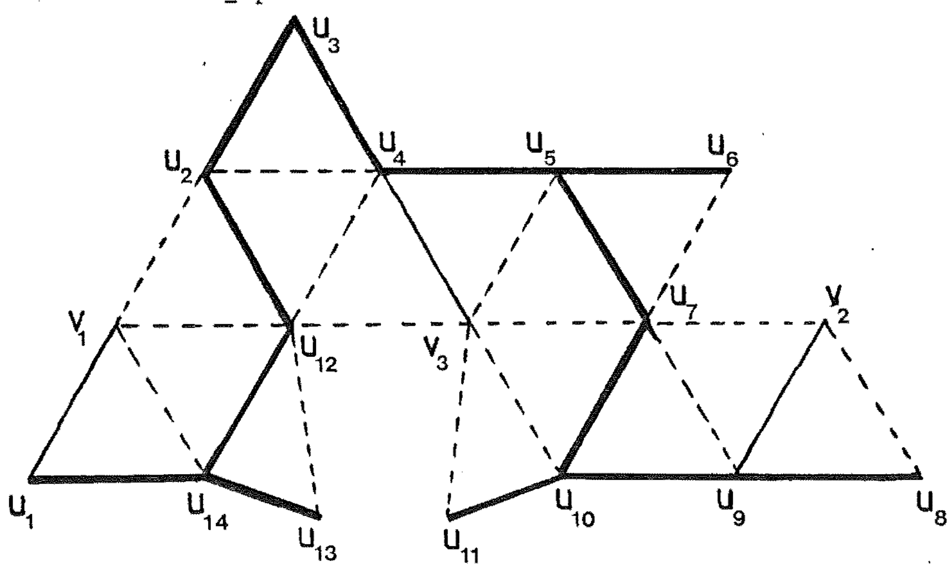


Figure 3.16 A maximal outerplanar graph G with spanning tree T. T is shown by solid lines. The linking subtree D of T induced by the six vertices of degree 2 in G is shown by heavier lines.

Theorem 3.28 Let T be a tree with n vertices. Then T can be embedded as a spanning tree of some maximal outerplanar graph G with exactly $\epsilon(T) + 2$ vertices of degree 2.

Proof: Let S be the linking subtree of T induced by the vertices x with $\beta(x) \geq 3$. All terminal vertices in S thus have $\beta(x) \geq 3$.

It may be possible to embed T in a maximal outerplanar graph in more than one way.

The first part of this proof makes T a labelled tree. The second part describes an embedding of this labelled tree in a labelled maximal outerplanar graph.

Part I

Let S have q vertices, where $q \geq 1$.

If $q = 1$ or 2, then label the vertices of S as Q_1 or Q_1 and Q_2 respectively.

If $q > 2$, perform a depth first search of the tree as in lemma 3.22, numbering the vertices $Q_1, Q_2, Q_3, \ldots Q_q$ in order as they are visited. Redraw the tree as in lemma 3.22, so that if edges are added joining each Q_i to Q_{i+1}, for $i=1$ to $q-1$, and Q_q to Q_1, an outerplanar graph results.

Let S_1 be the sequence $Q_1, Q_2, \ldots, Q_q$.

Form two other sequences from S_1 as follows:-

Let i be an integer between 1 and $q-1$ inclusive. If Q_i is not adjacent to Q_{i+1} in S_1 then there is a path in S from Q_i to Q_{i+1}. Insert the vertices on this path, other than Q_i or Q_{i+1}, in the order they occur between Q_i and Q_{i+1} in S_1. Repeat for every value of i. Also add any vertices on the path joining Q_q to Q_1, after Q_q in S_1 if Q_q is not adjacent to Q_1 in S. Underline all inserted vertices. This results in a new sequence S_2.

A non-terminal vertex X in S occurs at least once as $\underline{X}$ in S_2. Delete all but one of the occurences of $\underline{X}$ from S_2. Repeat for every non-terminal vertex of S. The resulting sequence is called S_3.

Note that Q_1 is the first term in each sequence. See figure 3.17, which shows a tree S and the outerplanar graph, shown dotted, in which S can be embedded. Here the three associated sequences are:-

$$S_1 = Q_1, Q_2, Q_3, \ldots, Q_{11}$$

$$S_2 = Q_1, Q_2, \underline{Q}_1, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, \underline{Q}_7, \underline{Q}_6, Q_9, Q_{10}, \underline{Q}_9, \underline{Q}_6, \underline{Q}_5, \underline{Q}_4, \underline{Q}_3, Q_{11}, \underline{Q}_3$$

$$S_3 = Q_1, Q_2, \underline{Q}_1, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, \underline{Q}_7, Q_9, Q_{10}, \underline{Q}_9, \underline{Q}_6, \underline{Q}_5, \underline{Q}_4, Q_{11}, \underline{Q}_3.$$

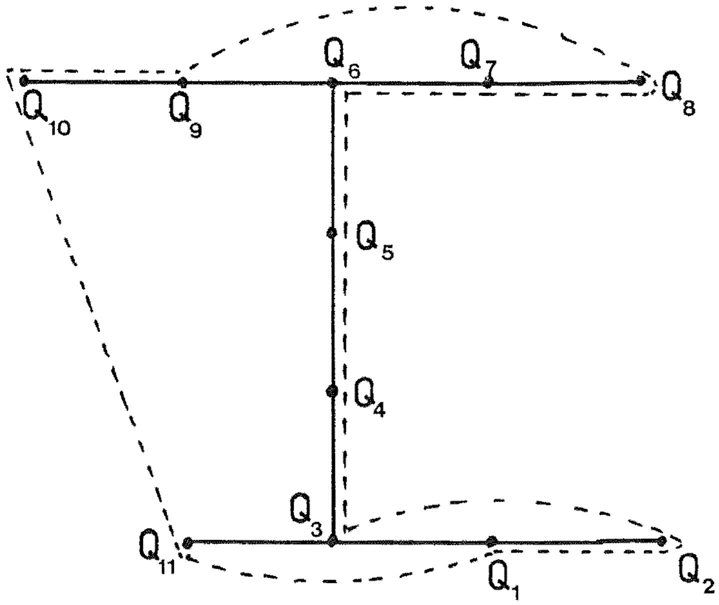


Figure 3.17 A tree S with the outerplanar graph in which it can be embedded shown dotted.

We now label the vertices that are in T but not in S.

Algorithm 3.2

Step 1: Let $i = 1$.

Step 2: If $deg_S(Q_i) = deg_T(Q_i)$ to to Step 4.

Consider the rooted subtrees of $T-Q_i$ which are rooted at vertices adjacent in T to Q_i but not lying in S.

Then $\beta(Q_i) - deg_S(Q_i) = k_i$ is the number of these subtrees which are branching. If $k_i = 0$ go to Step 3.

a. Let $\ell = 1$.

b. Consider one of the unlabelled rooted subtrees of T-Q_i described above. Denote this subtree of T-Q_i, containing n_ℓ-1 vertices where $n_\ell > 2$ as T_i^ℓ. Label the root u_{i1}^ℓ. The branching index of u_{i1}^ℓ in T could be 1, in which case let $m_\ell = 2$. Otherwise T_i^ℓ has m_ℓ-2, where $m_\ell > 2$, vertices with branching index 2 in T. From theorem 3.19 earlier, the subtree of T_i^ℓ induced by these vertices is a path graph (since every vertex x in T_i^ℓ has $\beta(x) \leq 2$ in T), with u_{i1}^ℓ as one of its terminal vertices, since one of the branching subtrees of T-u_{i1}^ℓ contains Q_i. If $m_\ell > 3$, label the m_ℓ-3 other vertices in T_i^ℓ with branching index 2 as u_{i2}^ℓ, $u_{i3}^\ell, \ldots, u_{im_\ell-2}^\ell$ in order as they occur in the path graph mentioned above, with u_{i2}^ℓ adjacent to u_{i1}^ℓ. Label the unlabelled neighbour of $u_{im_\ell-2}^\ell$ in T which is in a branching subtree of $T - u_{im_\ell-2}^\ell$ as $u_{im_\ell-1}^\ell$.

The remaining n_ℓ-m_ℓ vertices in T_i^ℓ are labelled v_{i1}^ℓ, $v_{i2}^\ell, \ldots, v_{in_\ell-m_\ell}^\ell$ using algorithm 3.1 in the sufficiency part of theorem 3.26 above.

c. If $\ell = k_i$ go to Step 3.

d. Increase ℓ by 1 and return to b.

Step 3: The remaining rooted subtrees of T-Q_i which are rooted at unlabelled vertices adjacent in T to Q_i, are rooted path graphs. For Q_i, there are $deg_T(Q_i) - \beta(Q_i)$ such subtrees. If $deg_T(Q_i) = \beta(Q_i)$ go to Step 4.

Consider the forest U, of these subtrees with p_i vertices. These vertices are labelled w_{i1}, w_{i2}, ..., w_{ip_i}, in order, as follows:

a. Consider a subtree of U, all of whose vertices are unlabelled of type (i) in figure 3.18. If none exist go to c.

Label the terminal vertex other than the root in this subtree. Continue labelling the unlabelled neighbour of the previously labelled vertex until the root is reached. Label the root.

b. Repeat a.

c. Consider a subtree of U, all of whose vertices are unlabelled of type (ii) in figure 3.18. If none exist, go to e. Label one of the terminal vertices of this tree. Continue labelling the unlabelled neighbour of the previously labelled vertex until the other terminal vertex is labelled.

d. Repeat c.

e. Consider an unlabelled subtree of U of type (iii) in figure 3.18. Label the root. If none exist, go to Step 4.

f. If there are still unlabelled vertices in U, return to e.

Step 4: If $i \neq q$, increase i by 1 and return to Step 2.

Note that Step 3 is similar to algorithm 3.1 earlier with u_k replaced by Q_i.

All vertices in tree T have now been labelled.

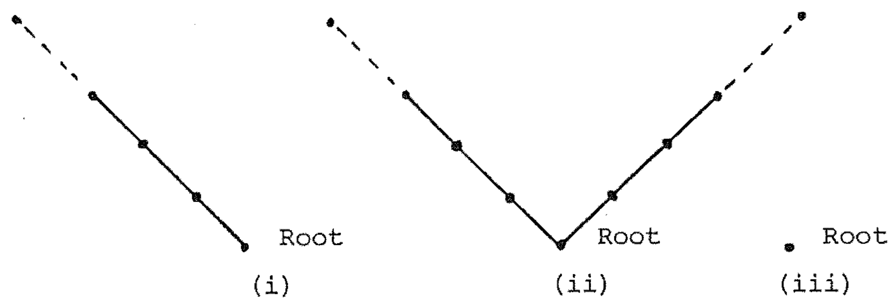


Figure 3.18 The three types of rooted path graphs with the root adjacent to Q_i in T.

Part II

The tree S was redrawn in Part I. Here we describe how to position the remaining vertices of T to give a particular embedding of T.

For each Q_i, if $deg_S(Q_i) = deg_T(Q_i)$ nothing can nor need be done.

However, if $k_i \neq 0$, then for each value of ℓ from 1 to k_i inclusive there is a subtree T_i^{ℓ} consisting of vertices $u_{i1}^{\ell}, u_{i2}^{\ell}, \ldots, u_{im_\ell-1}^{\ell}, v_{i1}^{\ell}, v_{i2}^{\ell}, \ldots, v_{in_\ell-m_\ell}^{\ell}$ each having branching index at most 2 in T. Also if $p_i \neq 0$, Q_i has p_i other vertices $w_{i1}, w_{i2}, \ldots, w_{ip_i}$ associated with it.

We have the following algorithm to embed these remaining vertices of T.

Algorithm 3.3

Step 1: Let $i = 1$.

Step 2: If $deg_S(Q_i) = deg_T(Q_i)$ go to Step 5.

a. If Q_i is terminal in S, then Q_i is adjacent in S to Q_t. Go to Step 3.

b. If Q_i is non terminal in S, then $\underline{Q}_i$ appears once in S_3, and at least once in S_2. Find the $\underline{Q}_i$ in S_2 corresponding to $\underline{Q}_i$ in S_3 with regard to the order of other vertices. Call this $\underline{Q}_i^2$. Let Q_h and Q_j be the vertices of S corresponding to the terms of S_2 before and after $\underline{Q}_i^2$ (or the first, Q_1, if $\underline{Q}_i^2$ is the last).

Step 3: a. If $k_i \neq 0$ and $p_i \neq 0$, position the vertices of T associated with Q_i, so that

(i) for each value of ℓ from 1 to $k_i, Q_i, u_{i1}^{\ell}, u_{i2}^{\ell}, \ldots, u_{im_\ell-1}^{\ell}$ are collinear and in order along the line, and $Q_i, v_{i1}^{\ell}, v_{i2}^{\ell}, \ldots, v_{in_\ell-m_l}^{\ell}$ are collinear also in order along the line, and

(ii) as one moves around Q_i anticlockwise, either the vertices $u_{i1}^1, v_{i1}^1, u_{i1}^2, v_{i1}^2, \ldots, u_{i1}^{k_i}, v_{i1}^{k_i}, w_{i1}, w_{i2}, \ldots, w_{ip_i}, Q_t$ if Q_i is terminal in S, or else the vertices $Q_h, u_{i1}^1, v_{i1}^1, u_{i1}^2, \ldots, u_{i1}^{k_i}, v_{i1}^{k_i}, w_{i1}, w_2, \ldots, w_{ip_i}, Q_j$ appear in cyclic order around Q_i, where Q_h, Q_j and Q_t are found as in Step 2 a. and Step 2 b. above.

b. If $k_i \neq 0$ and $p_i \neq 0$, position the vertices of T associated with Q_i as in (a) but with $w_{i1}, w_{i2}, \ldots, w_{ip_i}$ deleted from the sequences of vertices in (ii).

c. If $k = 0$ and $p_i \neq 0$, position the vertices $w_{i1}, w_{i2}, \ldots, w_{ip_i}$, as in (ii) of (a) above replacing the given sequences by $w_{i1}, w_{i2}, \ldots, w_{ip_i}, Q_t$ or $Q_h, w_{i1}, w_{i2}, \ldots, w_{ip_i}, Q_j$ respectively.

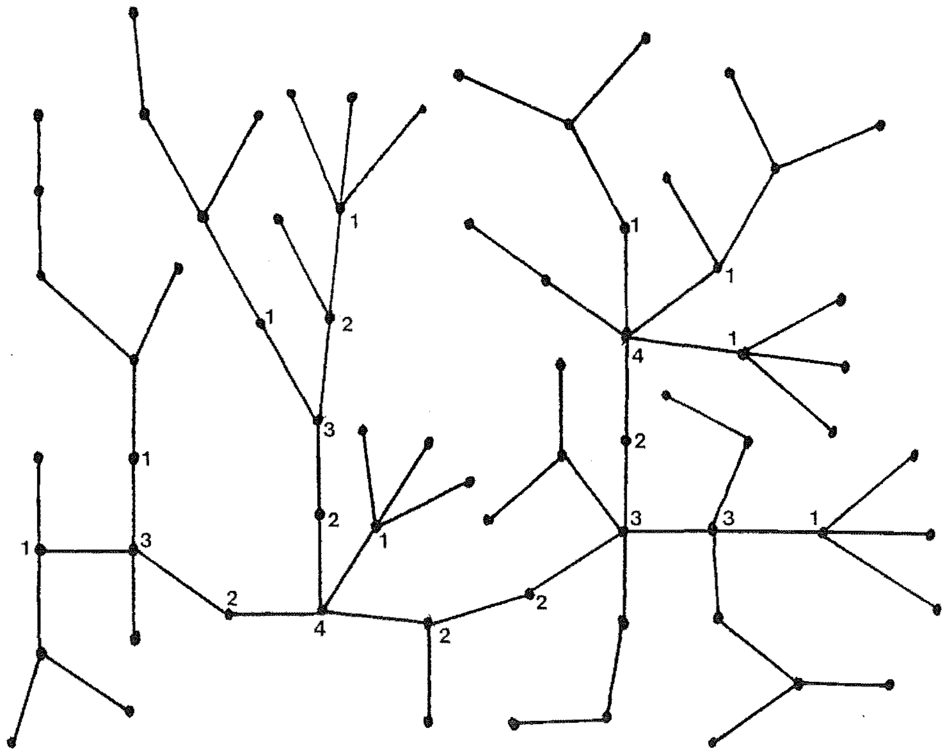
Step 4: If $i \neq q$ increase i by 1 and return to Step 2.

Step 5: All vertices of T are now positioned, so draw in the remaining edges of T. T has now been redrawn.

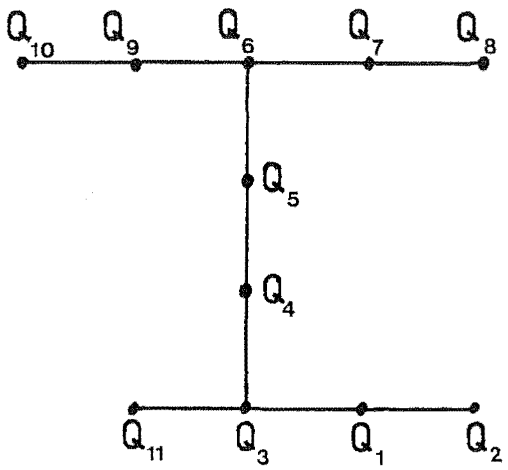
This reposition of vertices in T involved the swapping of the cyclic order of "arms" of T around the vertices, giving a particular embedding of T.

See figures 3.19 and 3.20. Note that the linking subtree S is that shown earlier in figure 3.17. The vertices Q_9, u_{61}^1, v_{61}^1, Q_5 appear in cyclic order anticlockwise around Q_6 as vertices Q_9 and Q_5 are before and after Q_6 in S_3. Vertices v_{61}^1, v_{62}^1, v_{63}^1 are collinear.

We now add edges to T where necessary, as described in the following algorithm, to create a block outerplanar graph O. Each cut vertex of O corresponds to some vertex in S.



(a)



(b)

Figure 3.19 A tree (a) with the linking subtree S induced by vertices having $\beta > 2$ (b). Numbers in (a) are β values. Each unlabelled vertex has $\beta = 1$.

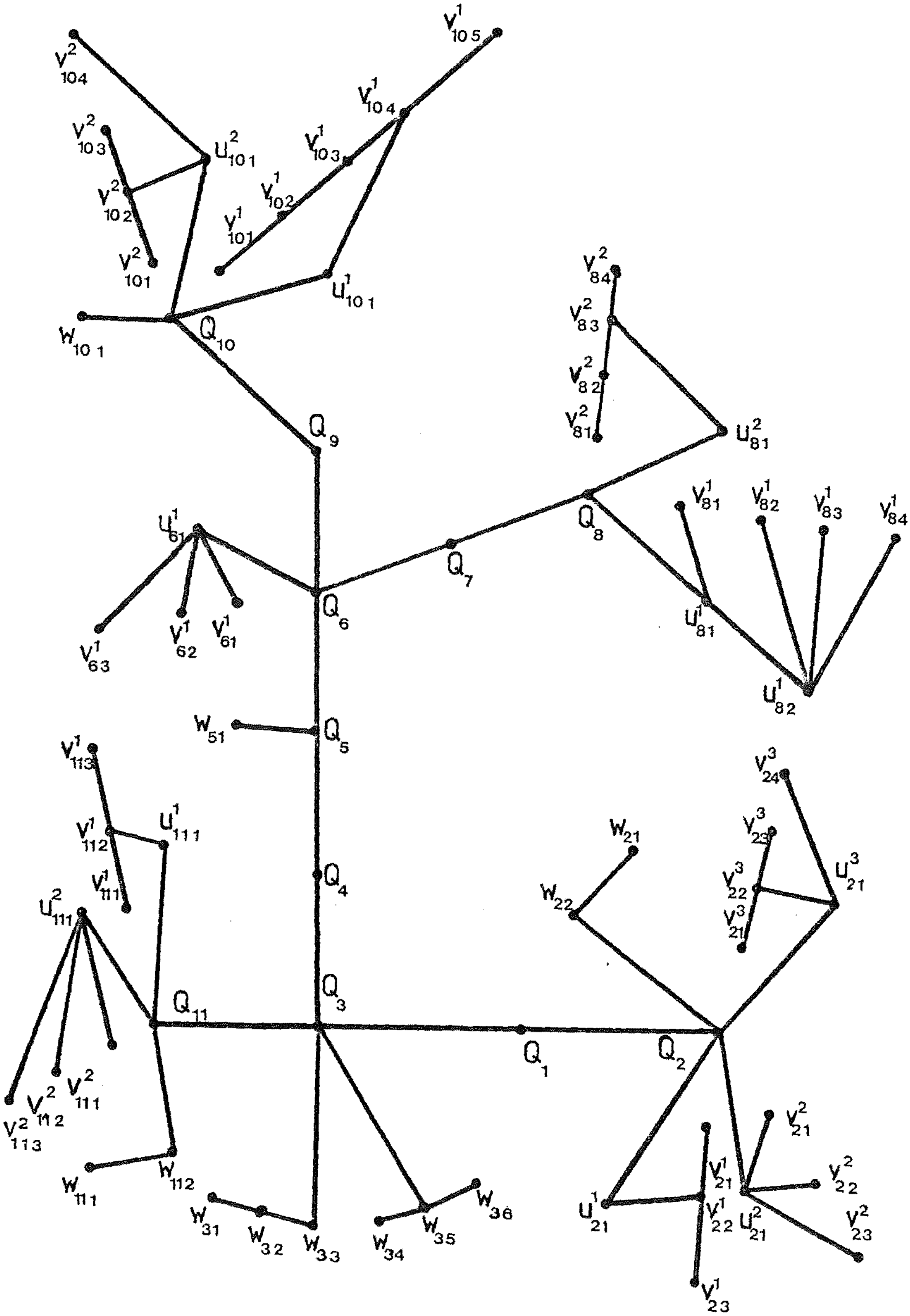


Figure 3.20 The embedding of the tree in figure 3.19(a) according to algorithm 3.3.

Algorithm 3.4 To form graph O from tree T

Step 1: Let $i = 0$.

Step 2: If $k_i \neq 0$, then for each value of ℓ from 1 to k_i,

a. join $u^{\ell}_{im_{\ell-1}}$ to $v^{\ell}_{in_{\ell}-m_{\ell}}$ if not already adjacent,

b. join v^{ℓ}_{i1} to v^{ℓ}_{i2}, v^{ℓ}_{i2} to $v^{\ell}_{i3}, \ldots,$ $v^{\ell}_{in_{\ell}-m_{\ell}-1}$ to $v^{\ell}_{in_{\ell}-m_{\ell}}$, and v^{ℓ}_{i1} to Q_i if not already adjacent.

(The subgraph of the graph so formed induced by vertices of T_i^{ℓ} and Q_i, is now outerplanar with hamiltonian circuit $Q_i, u^{\ell}_{i1}, u^{\ell}_{i2}, \ldots, u^{\ell}_{im_{\ell}-1}, v^{\ell}_{in_{\ell}-m_{\ell}}, \ldots, v^{\ell}_{i1}, \square_i$.)

c. For each u^{ℓ}_{ir} from $r=1$ to $m_{\ell}-1$, using algorithm 3.2 earlier, a set of vertices $\{v^{\ell}_{ip}, v^{\ell}_{ip+1}, \ldots, v^{\ell}_{ip+q}\}$ was labelled. Join each of these vertices to u^{ℓ}_{ir}, and join $v^{\ell}_{\mathrm{i}p+q}$ to u^{ℓ}_{ir+1} if $r \neq m_{\ell}-1$, if not already adjacent.

(So the subgraph mentioned in b. above is now a maximal outerplanar graph with exactly two vertices of degree 2, namely Q_i and $v^{\ell}_{in_{\ell}-m_{\ell}}$.)

Step 3: If $p_i > 1$ add edges to O joining each w_{ip} to $W_{i_{p+1}}$, and each w_{ip} to Q_i, for $p=1$ to p_i-1, and join w_{ip_i} to Q_i, if not already adjacent.

Step 4: If $i \neq q$, increase i by 1 and return to Step 2.

A new graph, O, has now been formed. Note that for each Q_i, provided $p_i > 1$, the subgraph of O induced by vertices $Q_i, w_{i1}, w_{i2}, \ldots, W_{ip_{\mathrm{i}}}$ is maximal outerplanar with exactly two vertices of degree 2, namely w_{i1} and w_{ip_i}. O is therefore block outerplanar.

The solid lines in figure 3.21 show the graph O formed in this manner from the tree in figure 3.20. Here $w_{11\ 2}$ is joined to w_{31} as Q_3 is the term in S_3 after Q_{11}. Also w_{22} is joined to Q_3, as both k_1 and p_1 equal zero for Q_1, the next term in S_3 after Q_2.

Edges are now added to form G, a maximal outerplanar graph using the following algorithm.

Algorithm 3.5 To change graph O into graph G

Step 1: For each vertex Q_i of S, if $k_i \neq 0$

join v_{i1}^{1} to u_{i1}^{2}, v_{i1}^{2} to $u_{i1}^{3}, \ldots, v_{i1}^{k_i-1}$ to $u_{i1}^{k_i}$,

and, if $p_i \neq 0$ join $v_{i1}^{k_i}$ to w_{i1}.

Step 2: Consider the first term of S_3.

a. This term is Q_i or $\underline{Q}_i$.

b. If it is Q_i corresponding to a non-terminal vertex in S, and is the last term in S_3 the algorithm ends.

c. If it is Q_i and corresponds to a non-terminal vertex in S, consider the next term in S_3, and return to a.

d. The term Q_i or $\underline{Q}_i$ corresponds to Q_i in S. If both $p_i = 0$ and $k_i = 0$, then provided the term is not the last, in which case the algorithm ends, consider the next term in S_3 and return to a.

(i) If $p_i \neq 0$, then

(α) consider the next term of S_3, or the first should Q_i or $\underline{Q}_i$ be the last. This term is Q_j or $\underline{Q}_j$.

(β) If the term is $\underline{Q}_j$, and

(1) $k_j \neq 0$, join w_{ip_i} to u_{j1}^{1}

or (2) $k_j = 0$ and $p_j \neq 0$, join w_{ip_i} to w_{j1}

or (3) $k_j = 0$ and $p_j = 0$, return to (α).

(τ) If the next term is Q_j, then join w_{ip_i} to Q_j

(ii) If $p_i = 0$ and $k_i \neq 0$, then do (α)(β)(τ) above with w_{ip_i}. replaced by $v_{i1}^{k_i}$.

e. Provided Q_i or $\underline{Q}_i$ is not the last term in S_3, consider the next term in S_3 and return to a.

<u>Step 3</u>: Any polygon which is not a triangle contains at least one of u_{i1}^{1}, v_{j1}^{k} or $w_{\ell p}$ for some values of i,j,k,ℓ and p. Choose one of these and join all vertices in the polygon to it if not already adjacent. Do this for every polygon in the graph until each polygon is a triangle.

The different components of O which contain Q_i, for each Q_i in S, are joined together in <u>Step 1</u>. In <u>Step 2</u> these are joined to components of O containing other vertices, using the bounding circuit of tree S in outerplanar graph H. Each w_{i1} and w_{ip_i} is joined to another vertex. An outerplanar graph is then formed as is shown in figure 3.21. <u>Step 3</u> triangulates each polygon forming a maximal outerplanar graph. It also ensures the degree of each Q_i in G is at least 3.

The vertices of degree 2 in G are given by $v_{in_\ell - m_\ell}^{\ell}$, for $\ell = 1$ to k_i, for each vertex Q_i of S. (See figure 3.22.)

For each Q_i, $k_i = \beta(Q_i) - deg_S(Q_i)$.

So
$$\begin{aligned}\sum_{Q_i \in S} (\beta(Q_i) - deg_S(Q_i)) &= \sum_{Q_i \in S} \beta(Q_i) - \sum_{Q_i \in S} deg_S(Q_i)\\ &= \sum_{Q_i \in S} \beta(Q_i) - 2(\sum_{Q_i \in S} 1 - 1)\\ &= \sum_{Q_i \in S} (\beta(Q_i) - 2) + 2\end{aligned}$$

is the number of vertices having degree 2 in G.

Since S is the linking subtree of T induced by vertices with $\beta > 2$ in T,

$$\sum_{Q_i \in S} (\beta(Q_i) - 2) = \sum_{u \in T : \beta(u) > 2} (\beta(u) - 2) = \epsilon(T).$$

The result of the theorem follows. #

The resulting maximal outerplanar graph G having the tree T shown in figure 3.19 as a spanning tree, is shown in figure 3.22. The branching index of T is eight, and the ten vertices of degree 2 in G are arrowed.

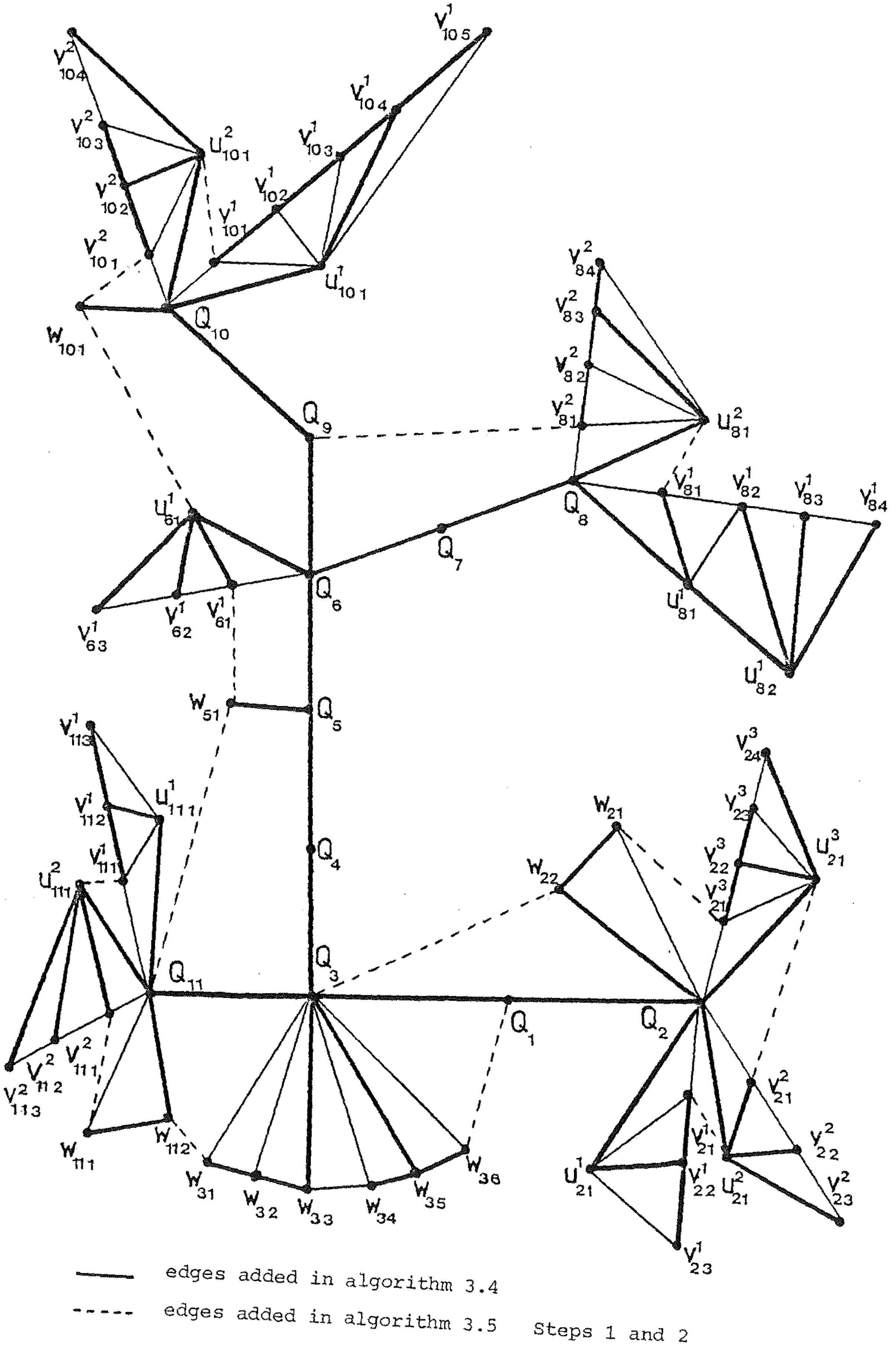


Figure 3.21 The outerplanar graph formed from T in figure 3.20.

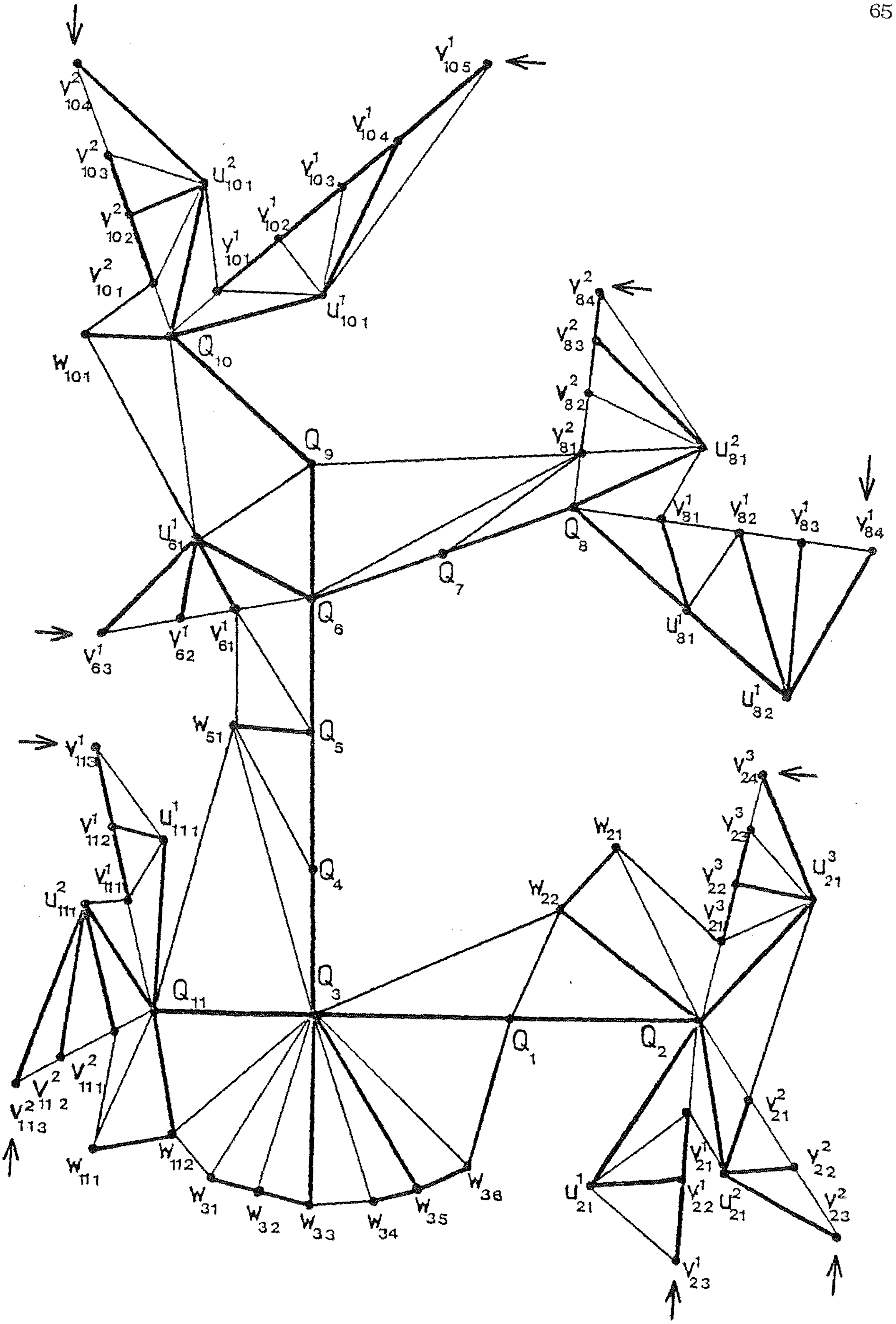


Figure 3.22 The maximal outerplanar graph G formed from the tree in figure 3.19. Degree 2 vertices are arrowed.

Corollary 3.29 A tree T can be embedded as the spanning tree of some maximal outerplanar graph G with exactly $\epsilon(T)+2$ vertices of degree 2, and $\epsilon(T)+2$ is the minimum number of degree 2 vertices any maximal outerplanar graph, for which T is a spanning tree, can have.

Proof: From theorems 3.27 and 3.28 above. #

CHAPTER IV

PROPER RECTANGULAR FLOORPLANS

From now on we are concerned with existence theorems for floorplans under given area and adjacency requirements. In this and the next chapter we assume a floorplan with given area is to be divided into a number of rooms, each with known area. Furthermore, we require each room to be external with its wall meeting the plan boundary in a particular way, and all joints in the plan to be 3-joints.

This chapter concentrates on rectangular floorplans.

I. DEFINING THE PROBLEMS

<u>Definition 4.1</u> A rectangular floorplan in which each room is external, is called an *exterior rectangular floorplan*.

Since floorplans with through rooms have adjacency graphs with multiple edges we restrict our attention to floorplans without through rooms. Thus each room meets the plan boundary in one continuous wall section. As seen earlier in chapter II, section II.A.3, a rectangular floorplan having 4-joints is a limiting case of a trivalent rectangular floorplan, that is, one in which only 3-joints occur.

<u>Definition 4.2</u> An exterior rectangular floorplan with no through rooms or 4-joints is a *proper rectangular floorplan*.

Since each room in a proper rectangular floorplan is adjacent to the exterior, we need only be concerned about the internal adjacencies, that is, those shown by the weak dual (see definition 2.9).

In this chapter we are interested in the following questions:

Problem A Which graphs are weak duals of proper rectangular floorplans?

Problem B Given a graph and specified areas for the vertices (rooms), can a proper rectangular floorplan be found whose weak dual is the given graph, also satisfying the area requirements?

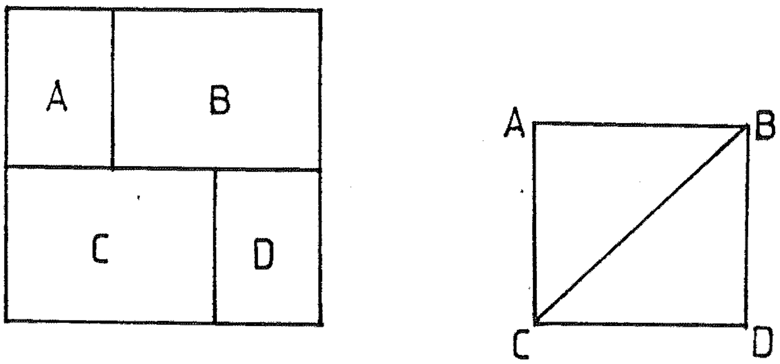
II. ADJACENCY GRAPHS OF PROPER RECTANGULAR FLOORPLANS

Lynes (1977) showed that the weak dual of a floorplan is outerplanar if and only if each room is external. If there are no through rooms and more than two rooms, the vertices of the weak dual lie on a bounding circuit. Removal of a single vertex will not disconnect the graph which is therefore two-connected. Further, if no 4-joints occur, the graph is maximal outerplanar. Vertices of degree 2 correspond to either corner rooms or endrooms. Clearly any maximal outerplanar adjacency graph with more than four vertices of degree 2 does not correspond to any rectangular floorplan (see Section I.B. of Chapter II). It follows from Earl and March (1979) and was shown by Syslo (1982) that every maximal outerplanar graph with at most four vertices of degree 2 is the weak dual of some proper rectangular floorplan. Thus we have the following theorem:

Theorem 4.1 An exterior rectangular floorplan with at least three rooms is a proper rectangular floorplan if and only if its weak dual is maximal outerplanar with at most four vertices of degree 2.

Figure 4.1 shows a rectangular floorplan with four rooms and its associated weak dual, which is maximal outerplanar. The corner rooms, A and D, correspond to vertices of degree 2 in the weak dual.

Recall from definition 2.14 that a fault line of a rectangular floorplan is a continuous straight run of adjacent internal walls between two joints on opposite sides of the plan boundary. A floorplan with a fault line can therefore be split into two other rectangular floorplans by a guillotine cut along the fault line.



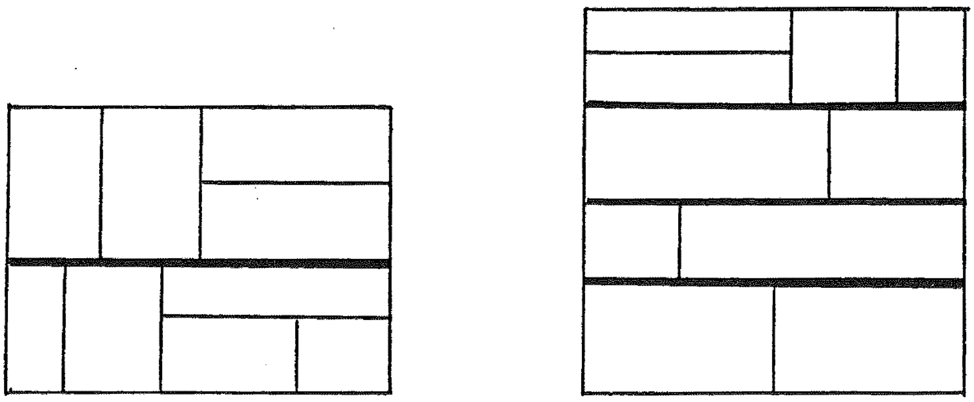
(a) (b)

Figure 4.1 A rectangular floorplan (a) and its maximal outerplanar weak dual (b).

<u>Theorem 4.2</u> Syslo (1982) Every proper rectangular floorplan has a fault line.

Examples of such fault lines are shown in figure 4.2.

The way in which the rooms meet the fault lines proves crucial for the area constraints. This is shown in the next section.



(a) (b)

Figure 4.2 Two proper rectangular floorplans with (a) one or (b) three horizontal fault lines.

Notation: We shall denote a room of a floorplan and the corresponding vertex in the weak dual by the same uppercase letter, and the area of the room by the corresponding lowercase letter. Consider the floorplan (a) shown in figure 4.3, with its accompanying weak dual (b), and grating (c) where x_1, x_2, x_3, y_1, y_2 are the dimensioning variables.

For this plan to be dimensioned to suit particular area values for rooms A, B, C, D while maintaining the same adjacencies, the length of the wall from α to β, or x_2 must be nonzero.

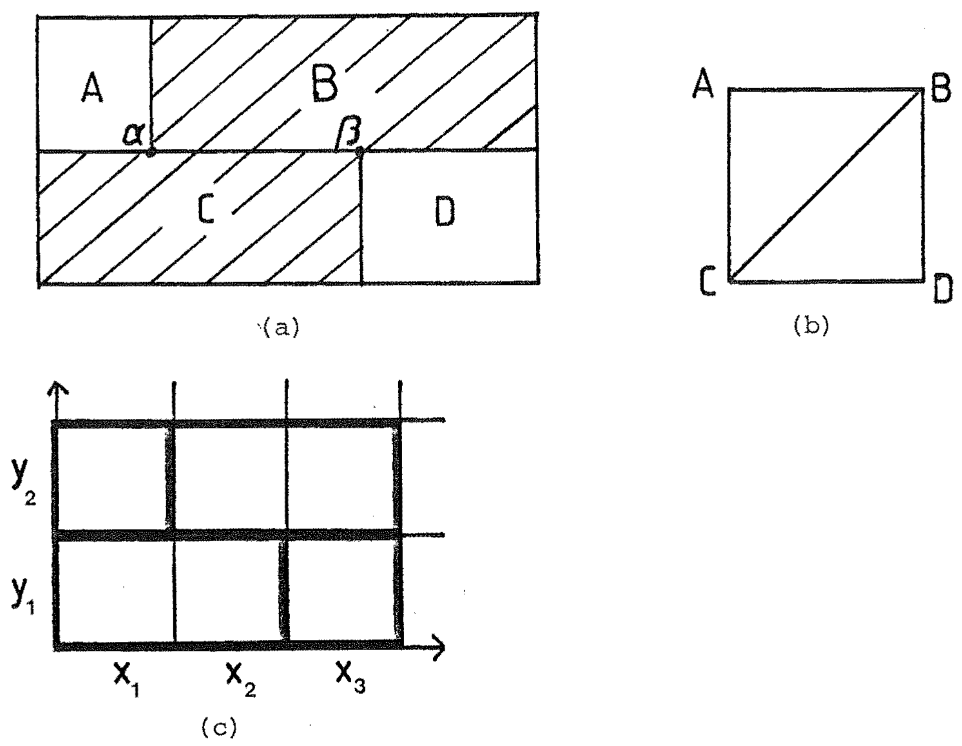


Figure 4.3 A floorplan (a) with its weak dual (b) and grating (c).

Theorem 4.3 In figure 4.3, $x_2 > 0$ if and only if $ad < bc$.

Proof: Since the areas a, b, c, d are nonzero, all of x_1, x_3, y_1, y_2 are strictly positive.

From the areas we have

$$x_1 \, y_2 = a \qquad (1)$$

$$(x_2 + x_3) y_2 = b \qquad (2)$$

$$(x_1+x_2)y_1 = c \qquad (3)$$

$$x_3\, y_1 = d \qquad (4)$$

So $\quad ad = x_1\, x_3\, y_1\, y_2 \qquad$ from (1),(4)

and $\quad bc = (x_1+x_2)(x_2+x_3)y_1y_2 \qquad$ from (2),(3)

$$= x_1x_3y_1y_2 + (x_1x_2+x_2^2+x_2x_3)y_1y_2$$

$$= ad + x_2^2y_1y_2 + x_2(x_1+x_3)y_1y_2$$

But y_1y_2 and x_1+x_3 are strictly positive, so $ad < bc$ if and only if $x_2 > 0$. #

Thus the product of the areas of the two unshaded regions in figure 4.3 is less than the product of the areas of the two shaded regions. A division of a rectangle into four rectangular regions adjacent in the manner of figure 4.3, has a similar area condition. In general, each area condition for any plan corresponds to one such division. For instance, in figure 4.4, only the wall section from α to β is crucial giving the condition $a(d+e) < bc$ as can be seen from (b). In (c) the wall section from α to γ gives the condition $ae < b(c+d)$, but this is weaker than the earlier one.

Each crucial condition in fact corresponds to an internal wall, which is one of the types shown in figure 4.5.

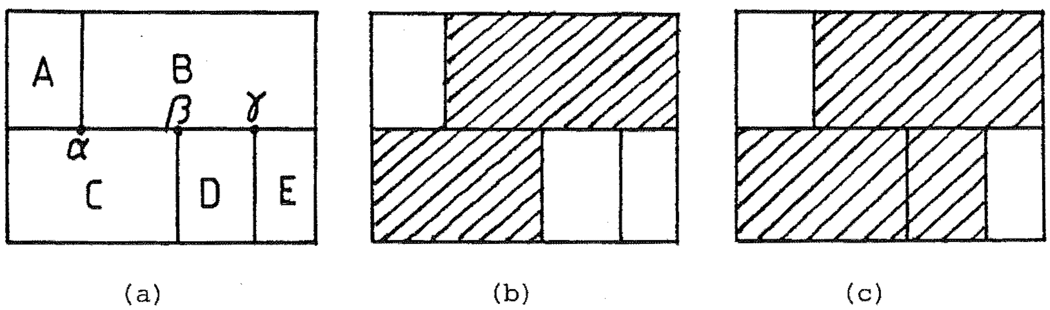


Figure 4.4 A floorplan (a) divided two different ways (b) and (c) to give area conditions.

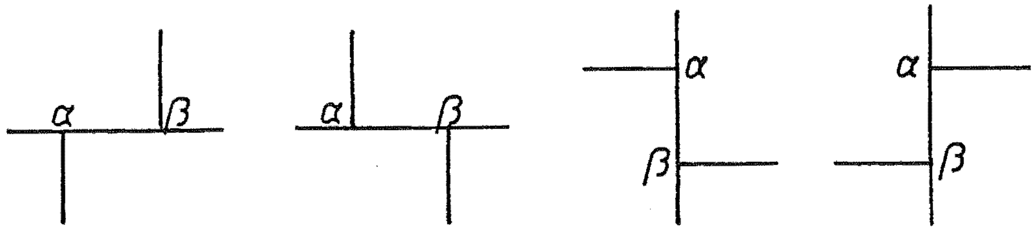
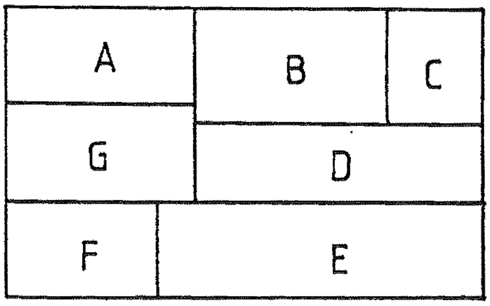
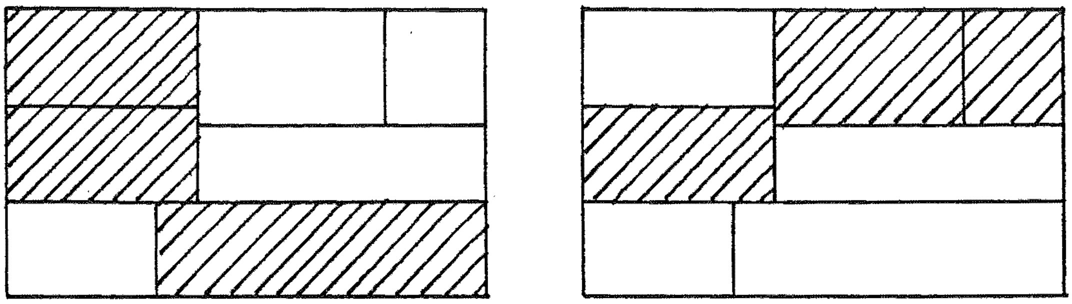


Figure 4.5 The four types of crucial internal walls.

Figure 4.6 is an example of a case where a smaller rectangle (of rooms A,B,C,D,G) in the plan gives rise to an area condition not involving all rooms in the plan.



(a)

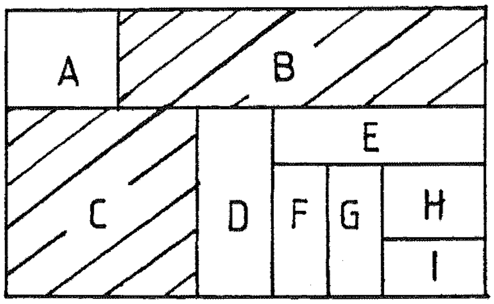


$f(b+c+d) < e(a+g)$ $ad < g(b+c)$

(b)

Figure 4.6 The two area conditions (b) for floorplan (a).

In figure 4.7 we have essentially the same situation as figure 4.4 except that corner room E has been further subdivided. However no more area conditions are created. This is of importance later.



a(d+e+f+g+h+i) < bc

Figure 4.7 A floorplan with its area condition.

IV TRANSFORMING A GRAPH INTO A PROPER RECTANGULAR FLOORPLAN

Given an maximal outerplanar graph G with at most four vertices of degree 2, a rectangular floorplan having G as its weak dual can be found. First, corner rooms and endrooms are chosen, where there are fewer than four vertices of degree 2. Then the edges of the graph are coloured in either of two 'colours' to specify the orientation of the corresponding walls, so that they satisfy the rules given in chapter II (page 22).

A floorplan having this coloured graph as its weak dual can then be derived. This can be done in several ways - for example, as outlined by Roth et al (1982).

However it can be done very simply by recalling from theorem 4.2 that the corresponding floorplan has a fault line.

Let the four corners of the floorplan correspond to the vertices *A*, *B*, *C*, *D* appearing in cyclic order around the exterior face or bounding circuit of the weak dual. Recall, from chapter II, that these may not be distinct, as an endroom corresponds to two consecutive vertices.

Draw a rectangle with the corners labelled *A*,*B*,*C*,*D* in a clockwise direction around it. If any vertices are repeated, so that two corners are labelled the same, delete vertices so that each vertex occurs once only in the perimeter of the rectangle. This perimeter can then be considered a graph. That is, it is a circuit including some of the vertices of the weak dual. Now add the remaining vertices of the weak dual in the appropriate places so that this circuit becomes isomorphic to the bounding circuit of the weak dual. Add in the remaining edges of the weak dual, and colour all edges as before to give a particular plane embedding of the coloured weak dual.

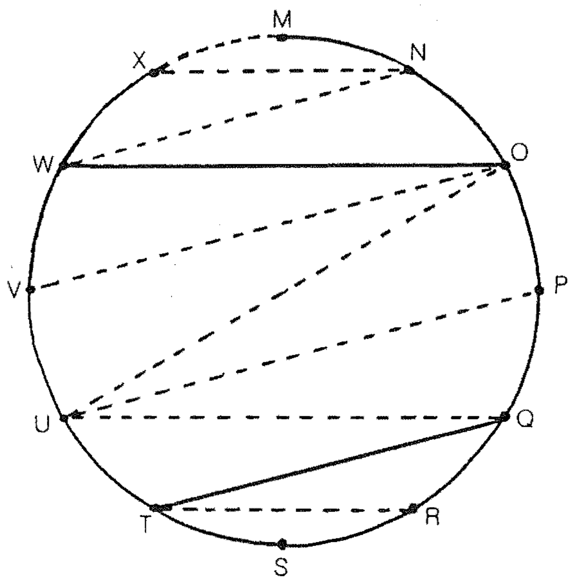
This is shown in figure 4.8. In (a) vertices X and M are chosen as corner rooms and S as an endroom. The graph is then coloured to satisfy the rules given in chapter II. The four corners correspond to vertices M, S, S, X around the bounding circuit. This is drawn as a rectangle in (b). In (c) one of the occurences of S is deleted and the remaining vertices of the weak dual inserted.

The floorplan is derived directly from this embedded weak dual. First, draw a rectangle enclosing the coloured weak dual with sides parallel to those of the bounding circuit. These sides form the plan boundary of a floorplan. It is possible to draw a line or lines parallel to one of the sides of the boundary across the floorplan so that the edges of the weak dual they cut across are all the same colour. Draw in these lines. (These correspond to fault lines in the final plan.) The original floorplan consisting of one rectangle (or room) has now been divided into several rectangles. Thus a new floorplan has been created.

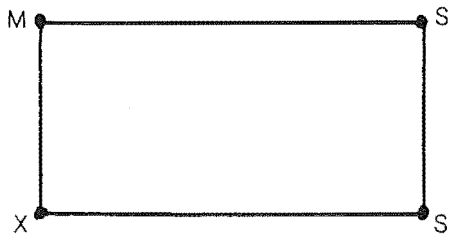
A new floorplan is continually created using the following steps:-

Step 1: Find the rectangles in the floorplan across which a line or lines can be drawn perpendicular to the direction in which the previous line or lines were drawn, cutting across only edges of the weak dual that have not already been cut and are all the same colour (opposite to the previous colour).

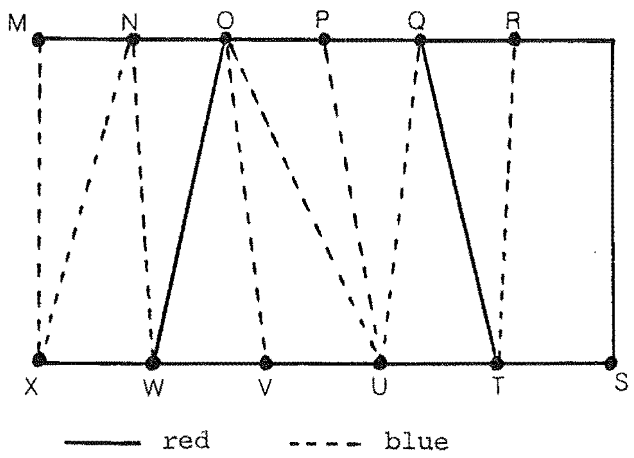
Step 2: Draw in these lines, ensuring the end points of each line create two new 3-joints in the plan. A new floorplan is then formed.



(a)



(b)



(c)

Figure 4.8 The redrawing (c) of a coloured maximal outerplanar graph (a) with corner rooms (b).

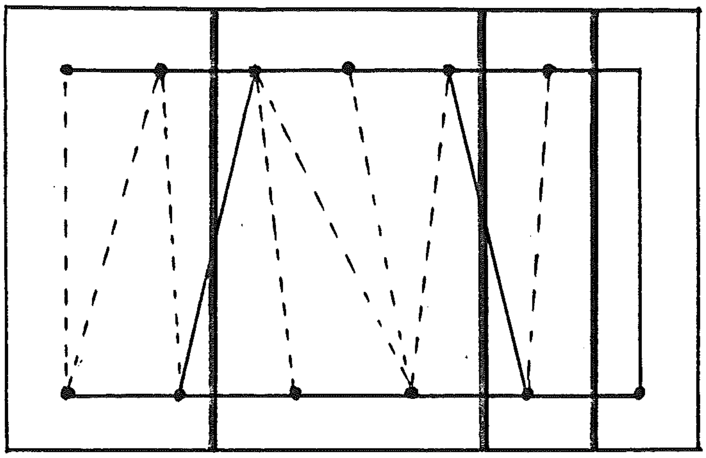
<u>Step 3</u>: If any edges of the weak dual have not yet been cut, return to step 1. Each edge of the weak dual is cut once, and the floorplan so formed has the given coloured graph as its weak dual.

This is shown in figure 4.9. In (a), the floorplan has vertical lines drawn across it, cutting only red edges of the weak dual. In (b) horizontal lines have been added in cutting blue edges only. Part (c) shows the completed floorplan.

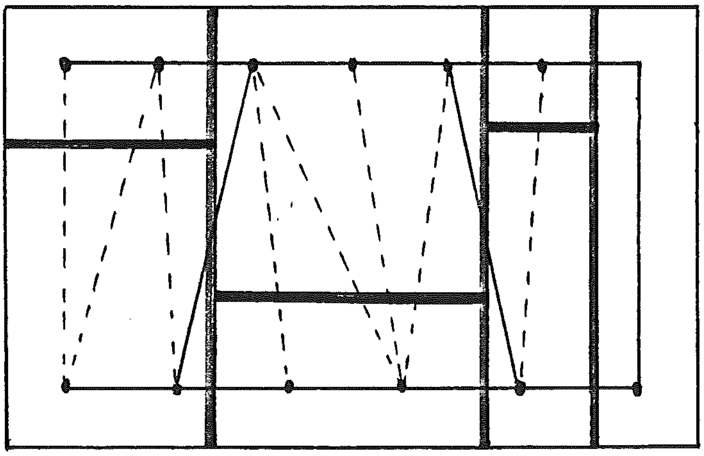
The necessary and sufficient conditions for the floorplan to be dimensioned correctly to satisfy any given area requirements can then be found.

Choosing corner rooms and endrooms, and colouring a given maximal outerplanar graph with at most four vertices of degree 2 in all possible ways, therefore allows all nonisomorphic undimensioned rectangular floorplans having the given graph as its weak dual to be found. Two 'colourings' of a weak dual differing in the colour of one edge correspond to two nonisomorphic floorplans sharing at least one common area condition.

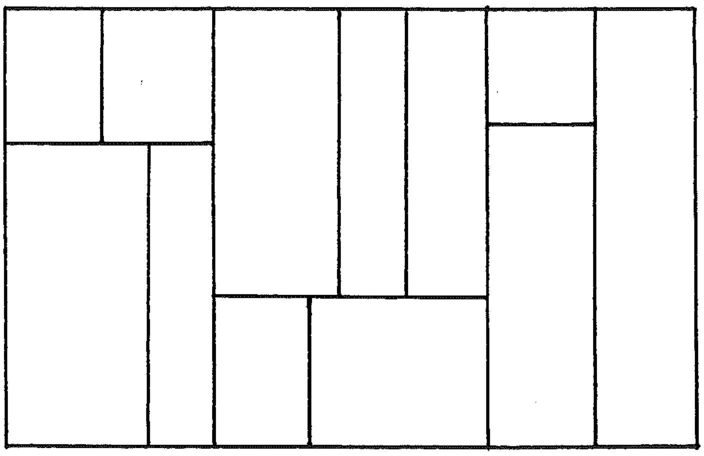
Figure 4.10 shows the nine nonisomorphic labelled rectangular floorplans (b) whose weak dual is given by (a). Two of these, namely (viii) and (ix) can only be found for a set of given area values if $ad < bc$.



(a)



(b)



(c)

Figure 4.9 A proper rectangular floorplan (c) having the coloured graph in figure 4.8(c) as its weak dual. The lines drawn in (a) and (b) cross only edges of the weak dual having the same colour.

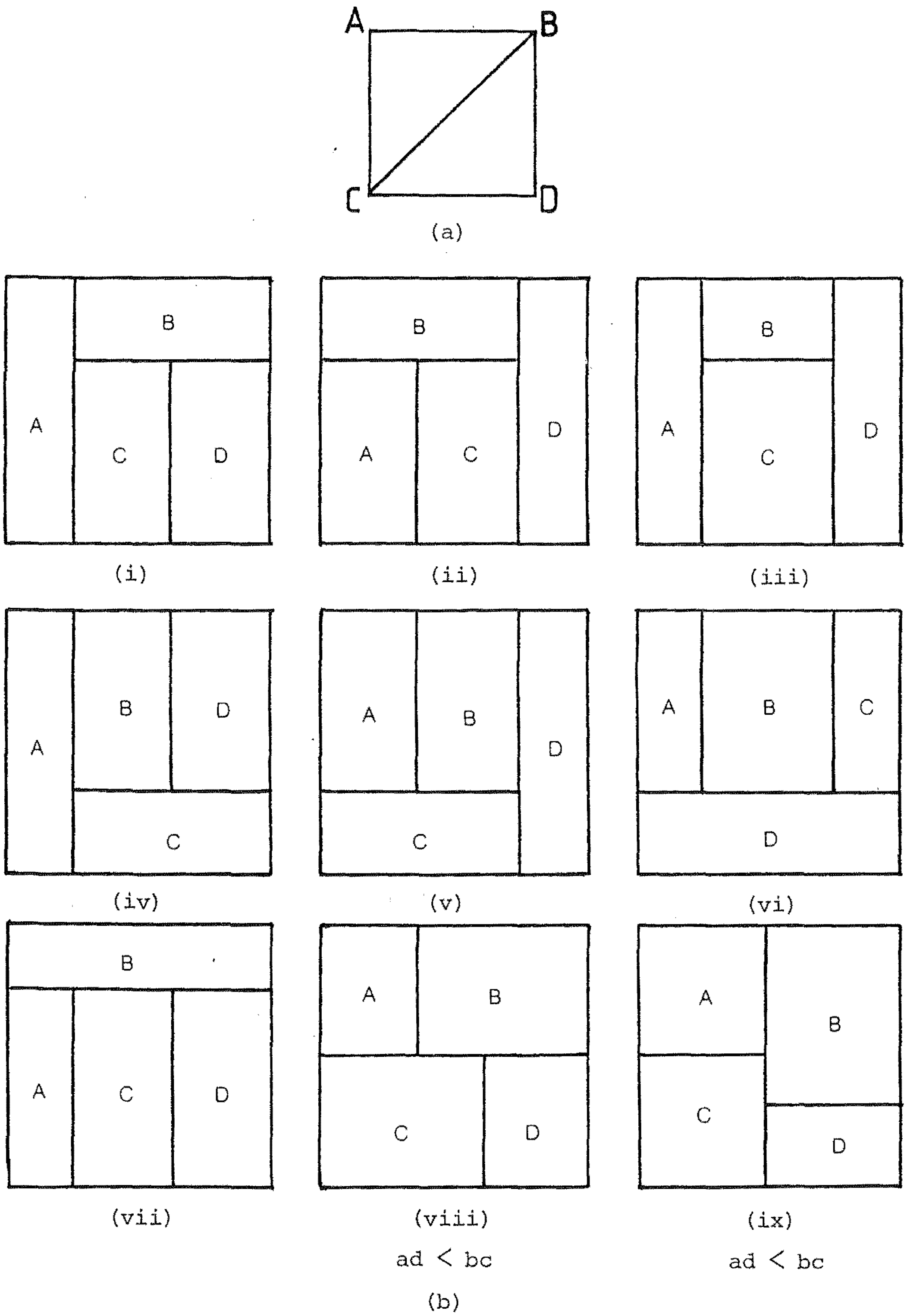


Figure 4.10 The nine nonisomorphic rectangular floorplans with their area conditions (b), having weak dual (a).

V COUNTER EXAMPLE FOR PROBLEM B

We now show by giving a counterexample that problem B is not always solvable.

Consider the maximal outerplanar graph shown in figure 4.11. As A,C,E and G all have degree 2, they must correspond to corner rooms.

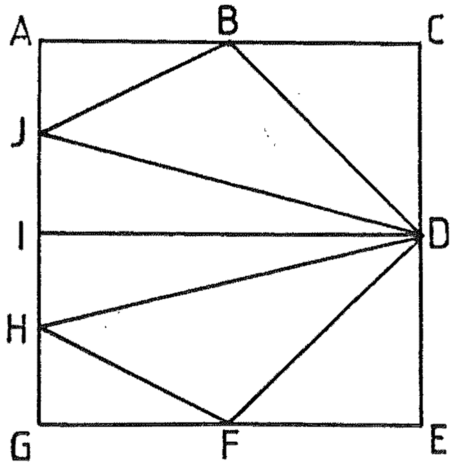


Figure 4.11 The adjacency graph for the counter example.

The sixteen nonisomorphic undimensioned rectangular floorplans with this weak dual are shown in figure 4.12. The necessary and sufficient conditions for each plan to be correctly dimensioned to suit particular area requirements are shown in Table 4.1. Part (b) shows which of inequalities 1-24 apply for each floorplan (i)-(xvi). Each inequality occurs twice in neighbouring plans.

Table 4.1. The necessary and sufficient conditions for a plan to be correctly dimensioned to suit particular area requirements. Part (b) shows which of the twenty-four inequalities in part (a) are required to dimension each of the sixteen plans from figure 4.12.

(a) Conditions

1	$(b+c)(f+g+h+i) < (a+j)(d+e)$	13	$(e+f)(a+i+j) < (b+c)(g+h)$
2	$e(h+i) < d(f+g)$	14	$c(f+g+h+i+j) < (a+b)(d+e)$
3	$(b+c)(h+i) < d(a+j)$	15	$e(h+i+j) < d(f+g)$
4	$dg < (e+f)(h+i)$	16	$c(h+i+j) < d(a+b)$
5	$(b+c)(g+h+i) < (a+j)(d+e+f)$	17	$dg < (e+f)(h+i+j)$
6	$i(e+f) < d(g+h)$	18	$c(g+h+i+j) < (a+b)(d+e+f)$
7	$i(b+c) < d(a+j)$	19	$(e+f)(i+j) < d(g+h)$
8	$a(d+e) < (b+c)(f+g+h+i+j)$	20	$ad < (b+c)(i+j)$
9	$e(a+h+i+j) < (f+g)(b+c+d)$	21	$c(i+j) < d(a+b)$
10	$ad < (b+c)(h+i+j)$	22	$e(a+b+h+i+j) < (f+g)(c+d)$
11	$g(b+c+d) < (e+f)(a+h+i+j)$	23	$(e+f)(a+b+i+j) < (c+d)(f+g)$
12	$a(d+e+f) < (b+c)(g+h+i+j)$	24	$g(c+d) < (e+f)(a+b+h+i+j)$

(b) Plan conditions

(i)	1,2	(vii)	10,15,16,17	(xii)	18,23,24
(ii)	1,8,9	(viii)	16,22,24	(xiii)	6,7
(iii)	8, 14, 15	(ix)	4, 5, 6	(xiv)	7, 13, 20
(iv)	14, 22	(x)	5, 11, 12, 13	(xv)	19, 20, 21
(v)	2, 3, 4	(xi)	12, 17, 18, 19	(xvi)	21, 23
(vi)	3, 9, 10, 11				

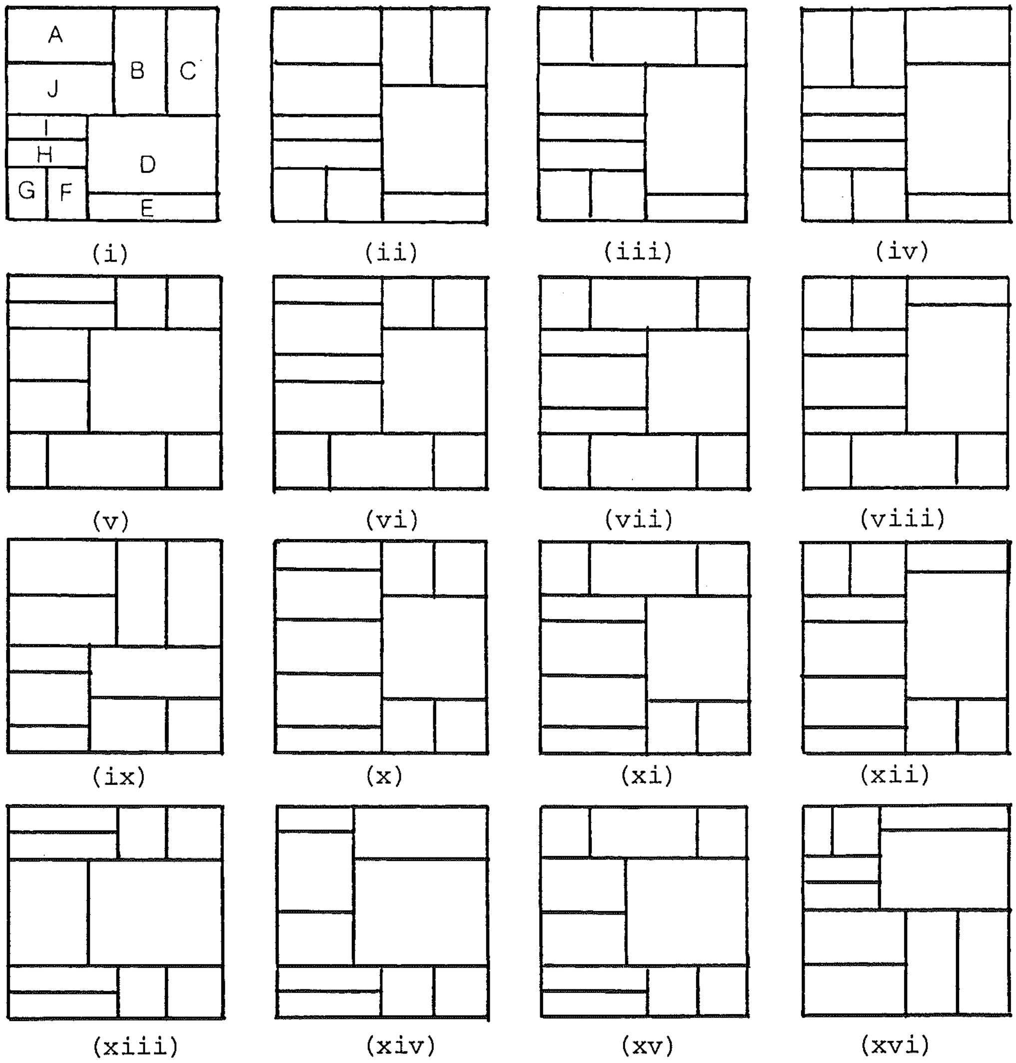


Figure 4.12 The sixteen nonisomorphic undimensioned floorplans with weak dual that in figure 4.11.

For each plan, at least one of the area conditions has i appearing on the left hand side of the inequality. Thus if i is sufficiently large each plan will be false. The same applies for c and e.

In fact if the areas of the rooms are

$a = b = c = d = e = f = g = 1$, $h = i = j = 2$, then conditions 1, 2, 3, 5, 6, 7, 9, 13, 14, 15, 16, 18, 19, 21, 22 and 23 are false, and at least one of the conditions for each plan will be false.

Thus we have proved:

Theorem 4.4 Given any maximal outerplanar graph with at most four vertices of degree 2, and areas associated with every vertex, the area of the corresponding room, it is not always possible to find a rectangular floorplan having the given graph as its weak dual and satisfying the area requirements.

VI FURTHER RESULTS

A. Other weak duals

1. Two vertices of degree 2

The only maximal outerplanar graph with five vertices is that shown in figure 4.13(a). Two of the 19 nonisomorphic undimensioned floorplans with their corresponding area conditions are shown in (b).

Since $ad > b(c+e)$

$\Rightarrow$ $be < ad$

$\Rightarrow$ $be < (a+c)d$,

at least one of these two plans can always be dimensioned for any given area values.

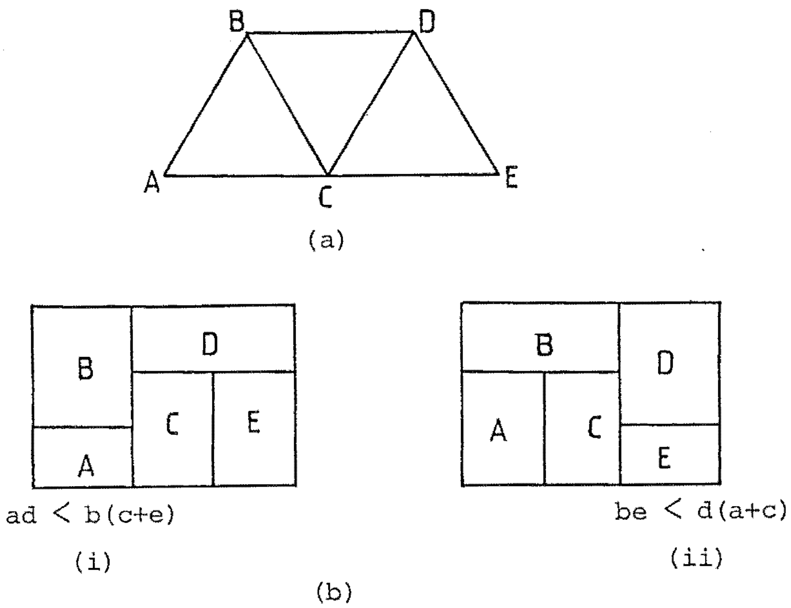


Figure 4.13 Two rectangular floorplans with their area conditions (b) having weak dual (a).

These two plans are important because any maximal outerplanar graph with exactly two vertices of degree 2 contains the graph in figure 4.13 as a subgraph.

Given a maximal outerplanar graph G with n vertices, where $n \geq 6$, but only two of degree 2, and given area values, label one of the degree 2 vertices A; its neighbours as B and C, D the vertex adjacent to B and C, and E the vertex adjacent to C and D. Label the remaining vertices $V_6, V_7, \ldots, V_n$ in increasing order choosing the next vertex to be labelled as that one adjacent to two already labelled vertices. Add the sum of the areas v_6, $v_7, v_8, \ldots, v_n$ to e, the area of E, giving a new value to e. Then either

(i) $be < (a+c)d$, or

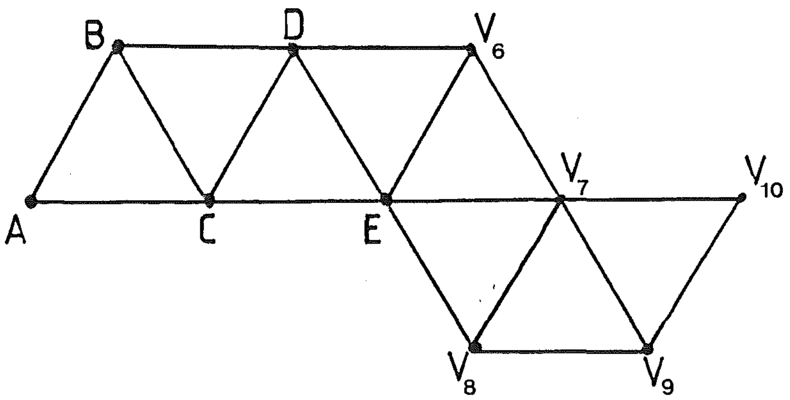
(ii) $ad < (c+e)b$.

Draw the corresponding floorplan, either (i) or (ii) of figure 4.13(b). The corner room E is then further divided into rooms $E, V_6, V_7, \ldots, V_n$. Vertex V_j is adjacent in G to two vertices - V_{j-1} (or E if $j = 6$) and another, already placed in the floorplan. If these two vertices meet across a vertical (or horizontal) wall, draw a line from a point on this wall horizontally (or vertically) across room V_{j-1} (or E), after deleting its label, to form two new rooms. Label the new corner room of the floorplan V_j, and the other room V_{j-1} (or E). Continue until V_n is positioned in the plan. This results in a floorplan which satisfies the adjacency requirements, and since the corner room E of figure 4.13 has been filled up in such a way not to create any new area constraints, it can be dimensioned to suit the given area conditions.

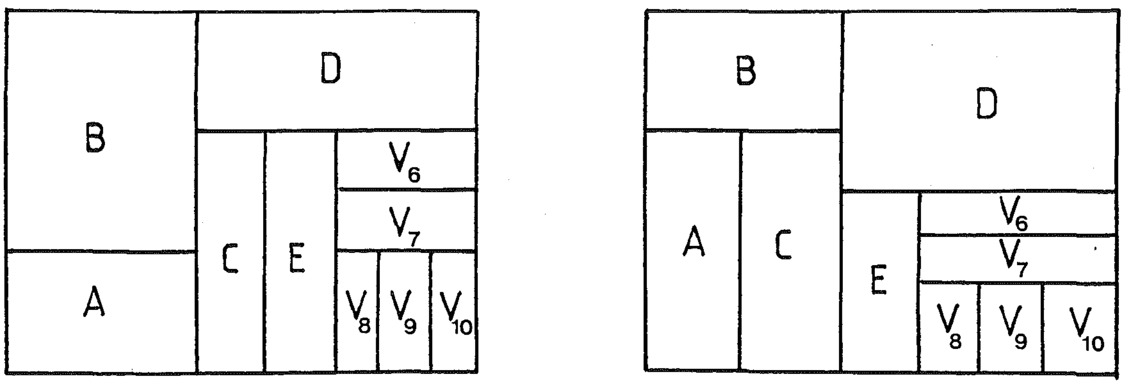
Thus we have the following theorem:

Theorem 4.5 Given any maximal outerplanar graph G with required areas for each vertex having only two vertices of degree 2, then a proper rectangular floorplan can always be found having G as its weak dual and satisfying the area requirements.

Figure 4.14 gives an example of this. One of the conditions in (b) will always be true for any given area values.



(a)



(i)

$ad < b(c+e+v_6+v_7+v_8+v_9+v_{10})$

(ii)

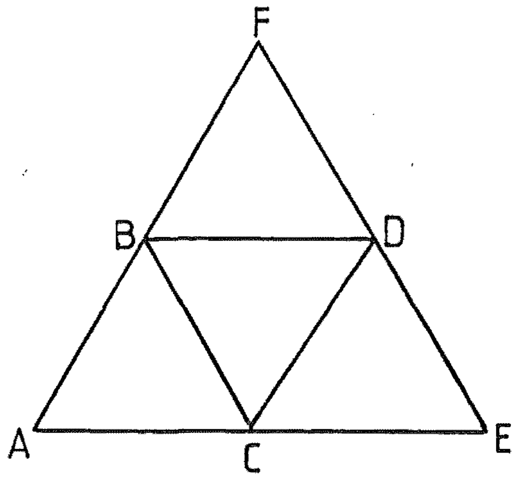
$b(e+v_6+v_7+v_8+v_9+v_{10}) < d(a+c)$

(b)

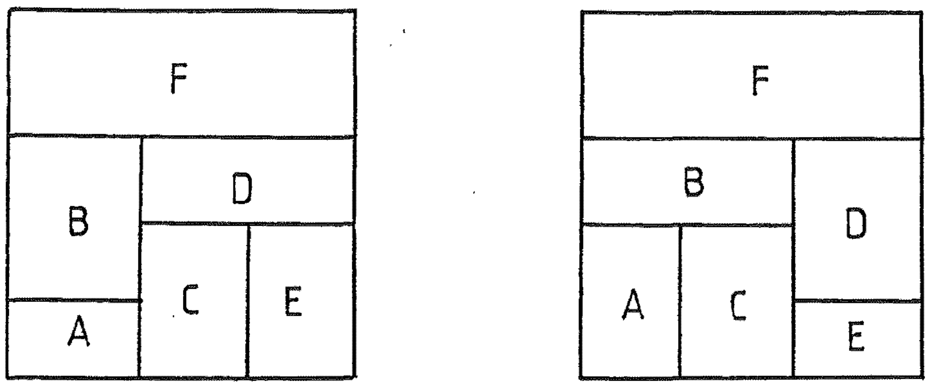
Figure 4.14 Two nonisomorphic floorplans with their area conditions (b) having weak dual (a).

2. Three vertices of degree two

The maximal outerplanar graph with three vertices of degree 2 having the least number of vertices is that shown in figure 4.15(a).



(a)



(b)

Figure 4.15 Two nonisomorphic floorplans (b) having weak dual (a).

Two of the rectangular floorplans having this graph as its weak dual are shown in (b). These are similar to those in figure 4.13(b) except that an extra room *F* has been added as an endroom. As this does not alter the area conditions, at least one floorplan can always be dimensioned to suit area and adjacency requirements of this graph.

Any maximal outerplanar graph with three vertices of degree 2 and more than seven vertices must contain a graph isomorphic to that shown in figure 4.16 as a subgraph.

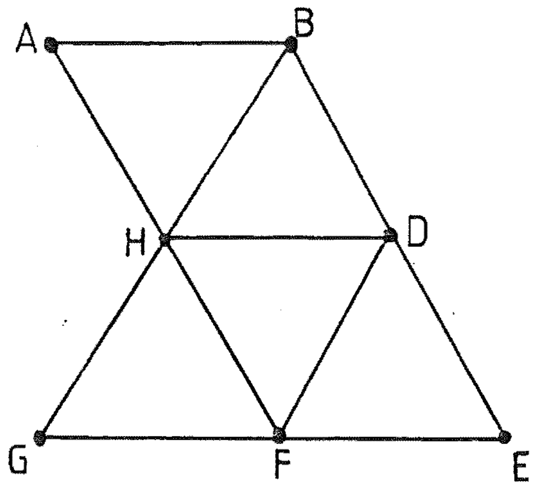


Figure 4.16 A maximal outerplanar graph with seven vertices, three of which have degree 2.

It is possible to prove the following theorem:

<u>Theorem 4.6</u> A proper rectangular floorplan can always be found having the graph in figure 4.16 as its weak dual, satisfying any area values and with corner rooms A, E, G and either B or D.

Given a maximal outerplanar graph G with n vertices, where $n > 7$, and three vertices of degree 2, and given area values, identify the subgraph of G isomorphic to that in figure 4.16 and label the vertices of this subgraph in the same way. The remaining vertices of G are labelled in the following way:-

<u>Step 1</u>: Is there an unlabelled vertex adjacent to A and one other labelled vertex in G? If not, proceed with <u>Step 4</u>.
Label this vertex U_1 and set $i = 1$.

<u>Step 2</u>: If U_i has degree 2 in G, proceed with <u>Step 4</u>. Otherwise label the unlabelled vertex of G adjacent to U_i and one other labelled vertex in G as U_{i+1}, and increase i by 1.

<u>Step 3</u>: Repeat <u>Step 2</u>.

<u>Step 4</u>: Repeat the above three steps replacing A by E, U by V, i by j, and <u>Step 4</u> by <u>Step 5</u>.

<u>Step 5</u>: Repeat <u>Steps 1</u> to <u>3</u> replacing A by G, U by W, i by k and <u>Step 4</u> by <u>Step 6</u>.

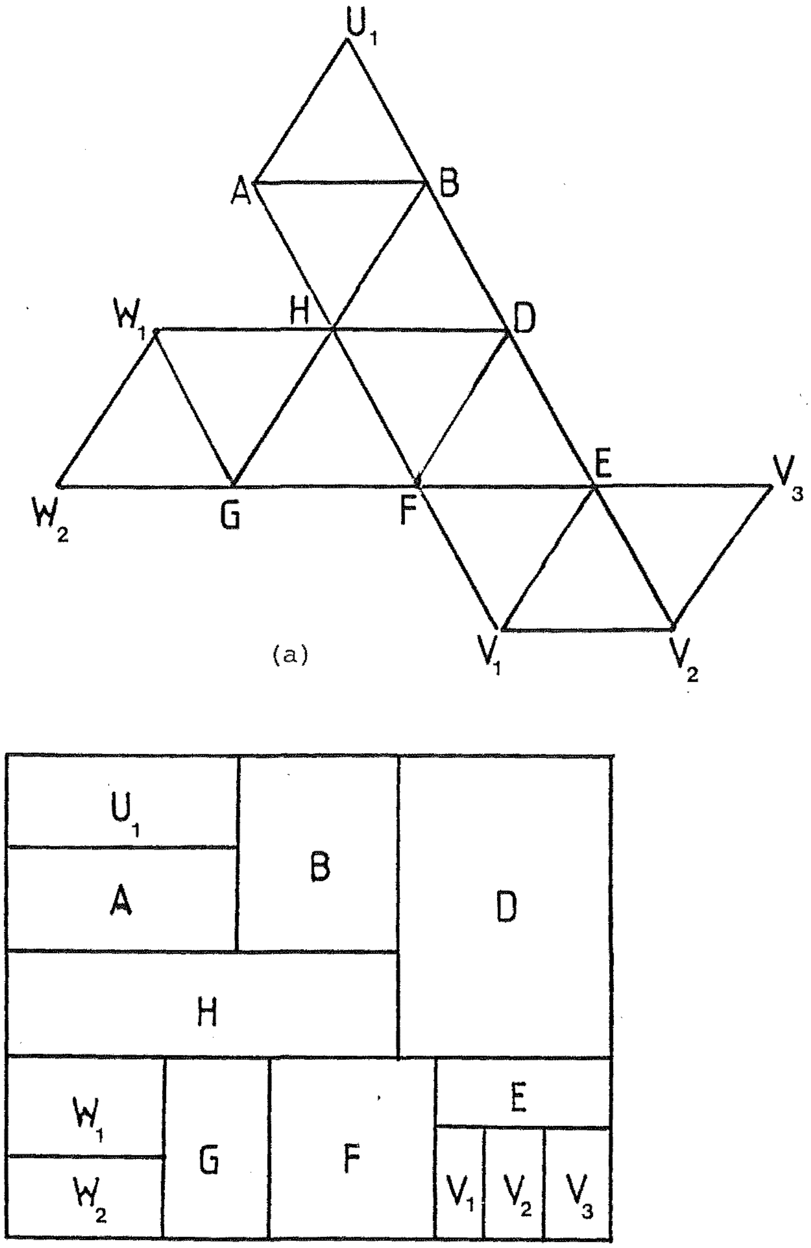
<u>Step 6</u>: Stop.

Add the sum of all the areas of type u_i, to a, the area of A. Similarly add the sum of all the areas of type v_j to e, and those of type w_k to g. This gives new values to a,e and g.

Using the area values a,b,d,e,f,g and h, draw a floorplan as mentioned in theorem 4.6 in which A,C,G and either B or D are corners, having the graph in figure 4.16 as its weak dual and satisfying the area conditions. The corner rooms A,C and G can then be divided, as E was divided in section 1 above for a maximal outerplanar graph with two vertices of degree 2, to form a rectangular floorplan which satisfies the area conditions and has the given graph G as its weak dual. An example of this is shown in figure 4.17. In (a) the vertices of G are labelled, and in (b) a corresponding floorplan with its area conditions is given. Note the positioning of rooms U_1,W_1,W_2,V_1,V_2 or V_3 does not create any new area conditions.

Thus we have the following theorem:-

<u>Theorem 4.7</u> Given a maximal outerplanar graph G with required areas for each vertex and having exactly three vertices of degree 2, then a proper rectangular floorplan can always be found having G as its weak dual and satisfying the area requirements.



(a)

$(u_1+a+b+h)(e+v_1+v_2+v_3) < d(f+g+w_1+w_2)$

$d(g+w_1+w_2) < (u_1+a+b+h)(e+f+v_1+v_2+v_3)$

(b)

Figure 4.17   A floorplan (b) with its area conditions having weak dual (a).

3. Four vertices of degree 2

The maximal outerplanar graph with four vertices of degree 2, having the least number of vertices (8) is shown in figure 4.18.

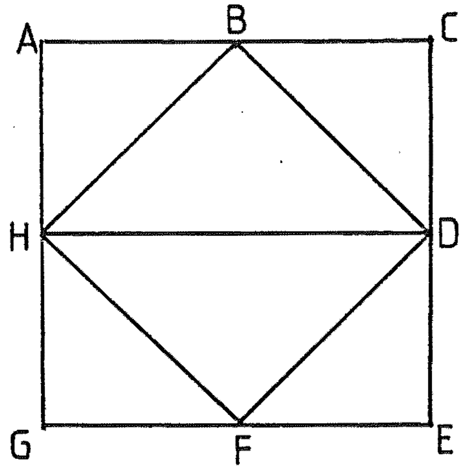


Figure 4.18 The maximal outerplanar graph with eight vertices, four of which have degree 2.

It can be shown that a proper rectangular floorplan can always be found having the graph in figure 4.8 as its weak dual, and satisfying any area requirements. Similarly a proper rectangular floorplan having any maximal outerplanar graph G with nine vertices, four of degree 2, as its weak dual and satisfying any given area values can always be found. Since the maximal outerplanar graph used in figure 4.11 for the counterexample to problem B in theorem 4.4 had ten vertices, of which four had degree 2, we have the following theorem:

Theorem 4.8 The graph used in figure 4.11 for the counterexample to problem B in theorem 4.4 is the maximal outerplanar graph G with the least number of vertices, for which no proper rectangular floorplan can always be found having G as its weak dual and simultaneously satisfying any given area requirements.

B. Proper rectangular floorplans with through rooms

So far in this chapter we have considered only rectangular floorplans without through rooms. It may appear that this is rather restrictive, but in fact often the opposite is true as through rooms limit the positioning of rooms between them.

For example, consider the graph G shown in figure 4.19.

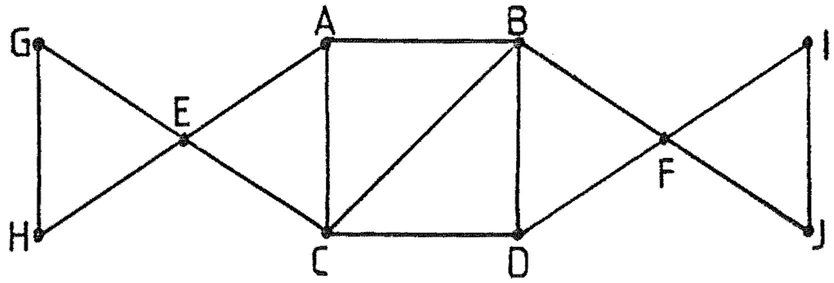
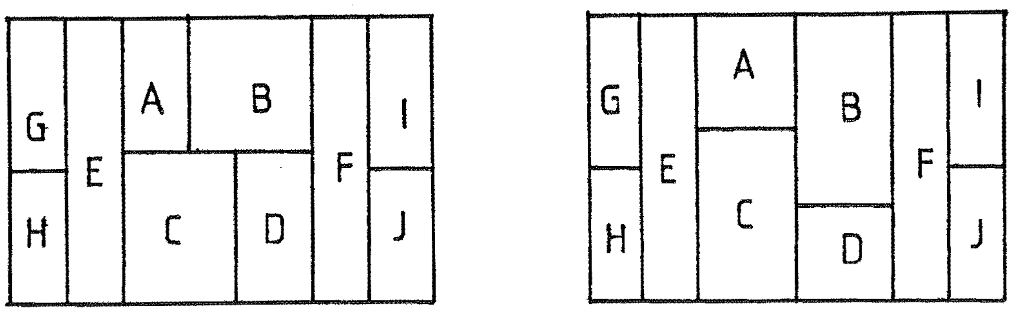


Figure 4.19 A block maximal outerplanar graph.

If this is to be the weak dual of a proper rectangular floorplan, then it follows from Syslo (1982) that *E* and *F*, the two cut vertices in the graph, must correspond to through rooms in the plan. Thus we have either of the two situations shown in figure 4.20. The subgraph of G induced by vertices *A*,*B*,*C* and *D* is that shown in figure 4.10 earlier. There were nine nonisomorphic rectangular floorplans for that graph, but here since *E* is required to be adjacent to both A and C, and *F* to *B* to *D*, only two of the nine plans are possible.



ad < bc   ad < bc

Figure 4.20 Proper rectangular floorplans having the graph in figure 4.19 as their weak dual.

In each, G and H, or I and J could be reversed without altering either adjacency or area conditions.

From this the following theorem can be concluded:

Theorem 4.9 Given a block maximal outerplanar graph G, with associated areas for each vertex, it is not always possible to find a proper rectangular floorplan having the given graph as its weak dual and satisfying the area conditions.

Proof: Consider the graph in figure 4.19. If this is to be the weak dual of some proper rectangular floorplan, then either of the situations in figure 4.20 must occur. Both require $ad < bc$, in order to be dimensioned correctly. Thus if we have areas for room A,B,C,D such that $ad > bc$, no proper rectangular floorplan suiting both area and adjacency requirements can be found. #

C. Number of nonisomorphic rectangular floorplans

Each rectangular floorplan corresponds to a partition of the edges of its weak dual G into two sets E_1 and E_2 so that when all the edges belonging to E_1 are coloured red (or blue) say, and those belonging to E_2 blue (or red), a valid colouring of graph G is obtained. See figure 4.21 where each edge in partition E_1 is coloured red.

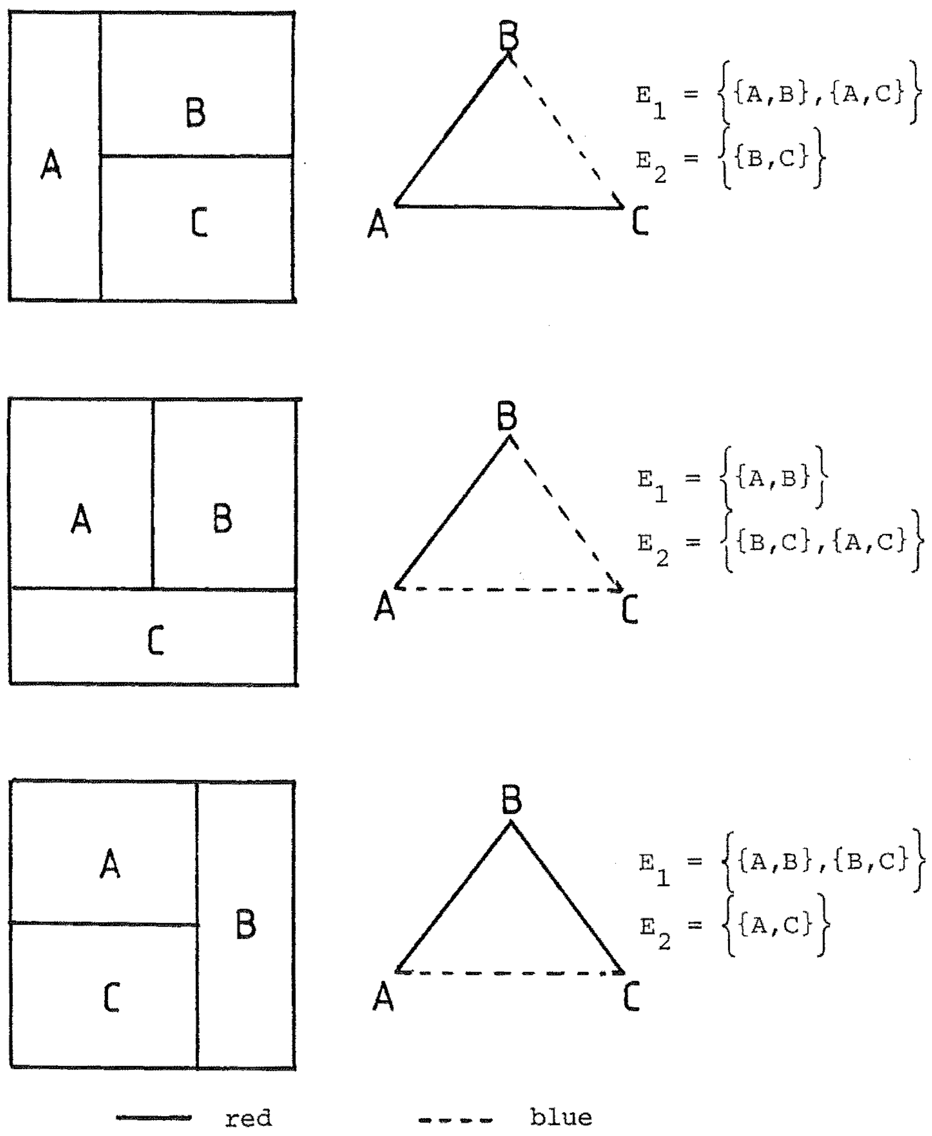
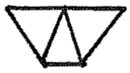
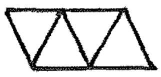
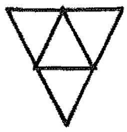
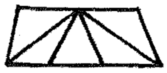
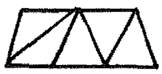
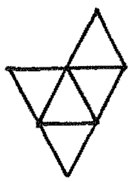


Figure 4.21 The nonisomorphic rectangular floorplans having K_3 as their weak dual. The partitions of the edges of the weak dual are shown on the right.

The number of nonisomorphic rectangular floorplans each having a given maximal outerplanar graph G as its weak dual, can be counted using an iterative formula dependent on the way in which G can be iteratively constructed from a triangle (see chapter II).

These are shown for graphs with at most seven vertices in Table 4.2.

Table 4.2 Nonisomorphic proper rectangular floorplans with $n \leq 7$ rooms.

Number of vertices (rooms)	Graph	Number of vertices of degree 2	Number of nonisomorphic proper rectangular floorplans
3		3	3
4		2	9
5		2	19
6		2	33
		2	37
		3	21
7		2	51
		2	59
		2	67
		3	29

From these, general results for particular types of graphs can be derived. For example, the number of nonisomorphic rectangular floorplans with n rooms, having weak dual of type shown in figure 4.22 can be found from the generating function $S(t) = (9-8t-2t^2+3t^3)(1-t)^{-2}(1-t-t^2)^{-1}$ Here the coefficient of t^i gives the number of nonisomorphic floorplans with i+4 rooms.

Figure 4.22 A particular type of maximal outerplanar graph.

CHAPTER V

ISOMETRIC AND CONVEX FLOORPLANS

In this chapter we extend problems *A* and *B* to more general floorplans. In a rectangular floorplan, all walls are parallel to one of two perpendicular directions. In the case of two non perpendicular directions, a shear transformation, area preserving, transforms the plan into a rectangular floorplan.

In the isometric case, there are three directions mutually at 120^o to each other, while in the convex case, there are a finite number of directions.

There are two cases to consider for problem B:- (a) boundary given and (b) boundary choosable within the above constraints. In the rectangular case the two are equivalent. For the isometric and convex cases we shall concentrate on the former.

I. ISOMETRIC FLOORPLANS

<u>Definition 5.1</u> The plan boundary and every room of an *isometric floorplan* are convex polygons with each wall parallel to one of three directions mutually at 120^o to each other.

In an isometric floorplan all angles between walls are 60^o or 120^o, so a grid of equilateral triangles can be imposed on the plan; hence the name isometric. All rooms have at most six walls, and any triangular is equilateral.

<u>Definition 5.2</u> A *proper isometric floorplan* is an isometric floorplan having only external rooms and 3-joints, and no through rooms.

Figure 5.1 illustrates one such proper isometric floorplan.

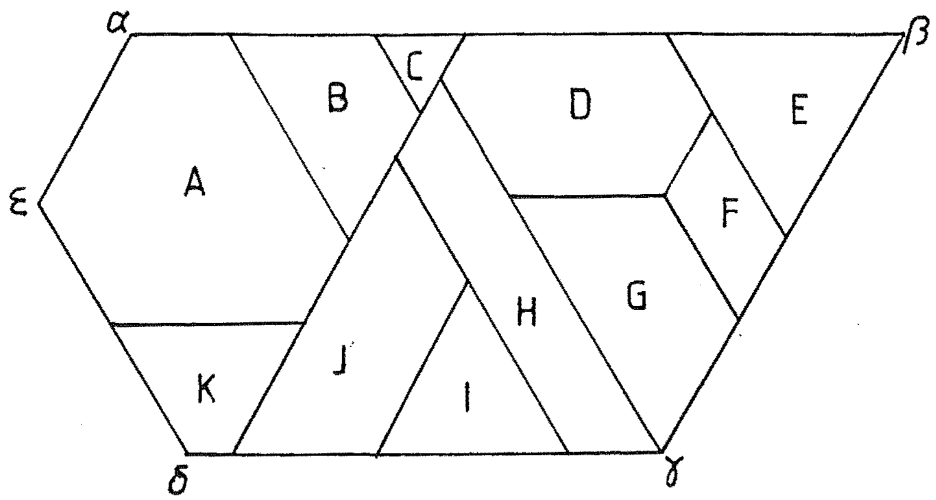


Figure 5.1 A proper isometric floorplan.

Definition 5.3 The points where the sides of the plan boundary meet are the *corners* of the plan. These may be also joints in the plan. A *corner room* has at least two walls on the plan boundary, and is further classified as an *n-corner room* where $n+1$ is the number of walls it has in common with the plan boundary.

Thus in figure 5.1, $\alpha,\beta,\gamma,\delta,\epsilon$ are the corners of the plan, and rooms A, E and K are corner rooms. As A has three of its walls on the plan boundary it is a 2-corner room. G and H are not corner rooms, and γ is a 3-joint.

A. Properties of proper isometric floorplans

A proper isometric floorplan has a weak dual which is maximal outerplanar. Vertices of degree 2 correspond either to a corner room or a room which is an equilateral triangle with one wall on the plan boundary. A 1-corner room having exactly two walls on the plan boundary is either a triangle, parallelogram or trapezium.

Theorem 5.1 Every maximal outerplanar graph is the weak dual of some proper isometric floorplan.

Proof: Given a maximal outerplanar graph G with a vertices, label the vertices 1 to n according to the constructive nature of the graph as outlined in theorem 3.11. That is, an initial triangle is labelled 1,2,3 and each i^{th} vertex for $i > 3$ is joined to two vertices having labels less than i.

Draw a convex polygon which has all walls lying in one of three directions mutually at 120° to each other. This is the plan boundary. Draw a line across the polygon parallel to one of the sides, and label the two rooms so formed as 1 and 2.

Add the remaining rooms in increasing order of the corresponding vertices as follows. Vertex i, for $i > 2$, is adjacent in G to two vertices j and k, where both $j < i$ and $j < k$. The two corresponding rooms are already in the plan and meet along a wall section. It may be that lines drawn at 60° degrees to this wall section on one or both sides meet other wall sections, but at least on one side, for points near enough to the boundary, the line so constructed will meet only the plan boundary. Choose an appropriate point and draw in one such line constructing a new room to be labelled i. Continue until all rooms have been added. As each room so constructed is convex, has each wall lying in one of the three given directions, and is adjacent to the exterior, the theorem holds. #

The proper isometric floorplan in figure 5.2(a) having the labelled graph in (b) as its weak dual has been constructed according to theorem 5.1. For example, prior to the placing of room 9, room 8 occupied the area taken up by rooms 8,9,10,11 and 12 in the final plan. As vertex 9 is adjacent in (b) to both 2 and 8, the point A on the wall section between rooms 2 and 8 was chosen, and the wall from A to B drawn at 60° to this wall section thereby forming the convex room 9.

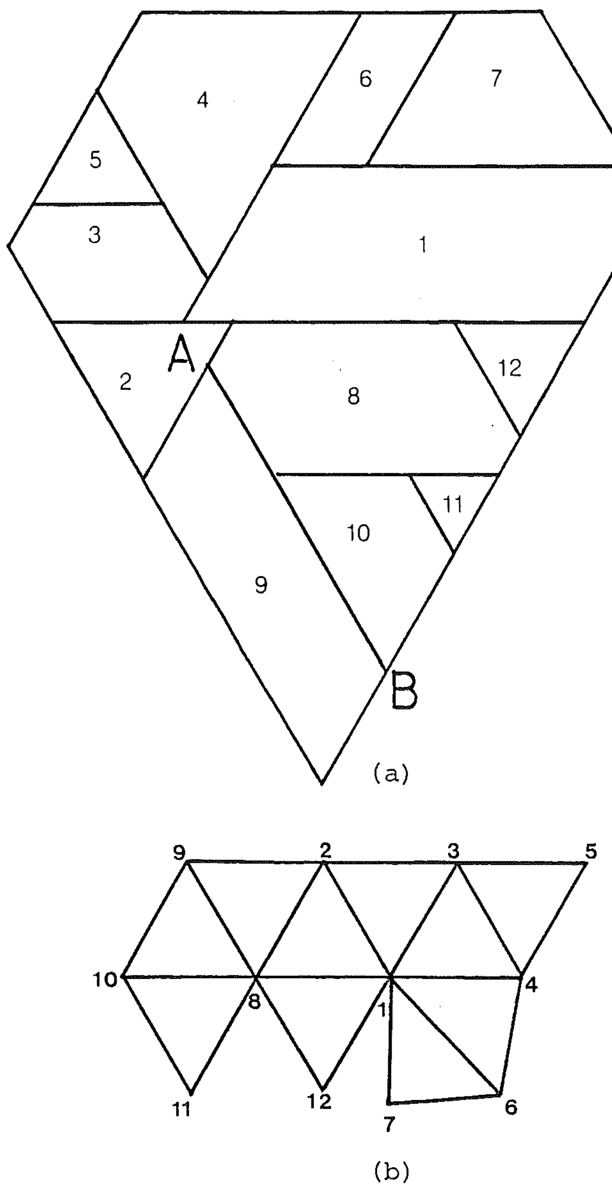


Figure 5.2   A proper isometric floorplan (a) with weak dual (b).

B. AREA PROBLEM

We now consider the analogous problem to problem *B* for isometric floorplans. That is,

<u>Problem C</u> Let *F* be any convex polygon whose walls lie in one of three directions mutually at 120° to each other, with area $A(F)$. Given a maximal outerplanar graph G with specified areas for each vertex, the sum of which equals $A(F)$, can *F* be divided to form a proper isometric floorplan having G as its weak dual and satisfying the area requirements?

Consider two equilateral triangles, with equal area, corresponding to vertices of degree 2 in G, along the same side of a proper isometric floorplan having G as its weak dual.

The situation is as shown in figure 5.3(a) where the walls of the rooms adjacent to A and C meet in one of the ways shown in (b).

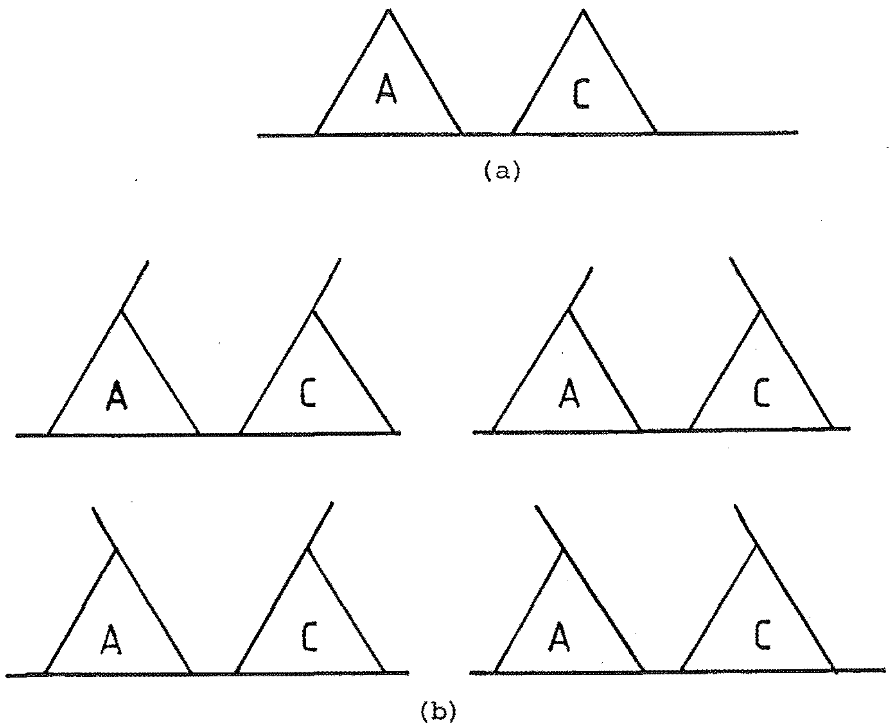


Figure 5.3 Two equilateral triangles with the same area having one edge in common with the same side of a proper isometric floorplan (a). Adjacent rooms have their walls meeting in one of the ways shown in (b).

Lemma 5.2 Consider a proper isometric floorplan whose weak dual G is maximal outerplanar. Let A,B,C be three non corner rooms of the plan having their external walls appearing consecutively along the same side of the plan boundary. Thus A,B and C occur in sequence in the bounding circuit of G. If A and C have equal areas, so that $a = c$, and correspond to vertices of degree 2 in G, then b, the area of B, must exceed a.

Proof: One of the situations in figure 5.3(b) occurs. Since B is the only room between A and C, the area of B must exceed the area of the polygon defined by vertices α,β,γ and δ, where α and δ are the 3-joints coincident with walls of rooms A and C not on the plan boundary, and β and γ the 3-joints coincident with the external wall of room B. See figure 5.4. Thus b must exceed a, the area of A. #

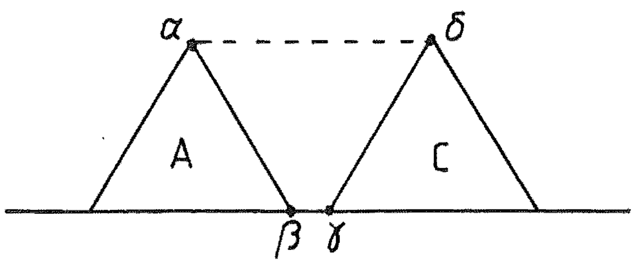


Figure 5.4 3-joints associated with rooms A and C in lemma 5.2.

Theorem 5.3 No proper isometric floorplan has a weak dual G, which is maximal outerplanar with 26 vertices, of which 13 have degree 2, as well as each room of equal area.

Proof: Since an isometric floorplan can have at most six boundary edges, six vertices in the graph can correspond to corner rooms.

Thus at least two vertices of degree 2 in the graph correspond to non corner rooms in the plan with external walls coincident with the same side σ of the plan boundary. From the properties of proper isometric floorplans earlier (section A), these vertices of degree 2 correspond to rooms which are equilateral triangles along σ.

Recall (corollary 3.10) that in a maximal outerplanar graph no two vertices of degree 2 are adjacent. Since G has 26 vertices, of which 13 have degree 2, there is exactly one vertex with degree not equal to 2 between any two with degree 2 in the bounding circuit of G. Let A and C be two such vertices of degree 2, with vertex B between them, in the bounding circuit of G, corresponding to non corner rooms along side σ. Then from lemma 5.2 the area of B must exceed the area of A, contradicting the assumption that all rooms have equal area. #

<u>Corollary 5.4</u> The graph in figure 5.5 is not the weak dual of any proper isometric floorplan in which all rooms have equal area.

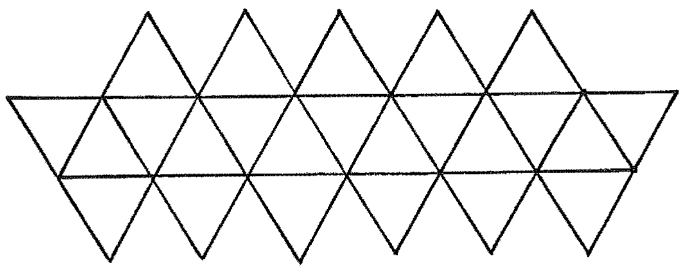


Figure 5.5 The graph for the counterexample.

Problem C was an extension of problem B to isometric floorplans with given boundaries - case (a) mentioned earlier (at the beginning of the chapter). In fact, corollary 5.4 shows problem B extended to isometric floorplans for either case (a) or (b) is not always possible.

<u>Remark</u>: Since the graph in figure 5.5 has more than four vertices of degree 2, it is also unable to be the weak dual of any proper rectangular floorplan.

II. CONVEX FLOORPLANS

Definition 5.4 Each room and the plan boundary of a *convex floorplan* are convex polygons.

Consider the following problem:

Problem D Given any maximal outerplanar graph G with required areas for every vertex, can a convex floorplan be found satisfying both area and adjacency requirements?

Robinson and Janjic (1985) showed this was always possible by describing more generally how any convex polygon could be divided to form a convex floorplan satisfying the area and adjacency requirements given by any 2-connected outerplanar graph.

Edges were added to the graph to form a maximal outerplanar graph, and then the constructive nature of maximal outerplanar graphs was used to assign levels, precursor and ancestors to the vertices. The areas were amalgamated by starting with a vertex X which has no successors, and adding its area to the area of its precursor, Y, and then repeated for vertex Y until a vertex of level 0 was reached. This continued until all the area was shared between the two level 0 vertices.

Initially a polygon with the correct area was divided by a straight line, into two convex polygons having the final amalgamated areas. Then each polygon was further divided by choosing a point or apex, from which a line was drawn to the plan boundary so that a new room Z, having precursors X and Y already in the plan, with the correct area was formed. Rules regarding the choice of apex to ensure each room was external, and that the plan had the given graph as its weak dual were given.

This is a generalization of the construction method outlined in theorem 5.1. Thus we have the following theorem.

<u>Theorem 5.5</u> A proper convex floorplan can always be found satisfying the area and weak dual adjacency requirements of any given maximal outerplanar graph.

Although there exist planar graphs with associated areas unable to be the weak duals of any proper rectangular or isometric floorplan, there may be planar connected graphs with associated areas which can always be realised as the weak duals of rectangular or isometric floorplans.

This is investigated in the next chapter.

CHAPTER VI

TREE ADJACENCY AND AREA REQUIREMENTS

Earlier it was stated that the adjacency requirement graph is connected and planar. A connected graph with n vertices having the least possible number of edges is a tree with $n-1$ edges.

In this chapter we consider the following question:

Tree adjacency problem Given a tree T representing required internal adjacencies between rooms and areas for each room, is it possible to form a rectangular, or isometric or convex floorplan so that both the required adjacencies and areas are satisfied? Adjacencies beyond those required by the tree are permitted.

It is shown that this is always possible. First an algorithm showing how these requirements can be satisfied for a particular type of convex polygon is given. The modifications of this algorithm for each type of floorplan are then given along with a more general algorithm.

I. PARABOLIC POLYGONS

Definition 6.1 A polygon having sides $\alpha_1,\alpha_2,\alpha_3$ in consecutive order around its perimeter with the sum of the interior angles between sides α_1 and α_2, and sides α_2 and α_3 being at most 180°, is a *parabolic polygon.*

Consider a parabolic polygon F with area $A(F)$. Let T be a tree with n vertices $A,B,\ldots,N$. Further let each vertex in the tree have an associated area $a,b,\ldots,n$ with the sum of the areas equalling $A(F)$.

Consider the following algorithm to divide F into n convex polygons labelled $A,B,\ldots,N$, thus forming a convex floorplan.

<u>Algorithm 6.1</u>   For a parabolic polygon.

<u>Step 1</u>: Select root and assign levels.

a. Change tree T into a rooted tree T_x by choosing any vertex X to be the root of the tree.

b. Assign a level to each vertex in T_x, where vertex Y has level k if there is a path of length k from the root X to Y.

<u>Step 2</u>: Assign a new area v^1 to each vertex V.

Begin with the highest level.

a. Take each vertex V of level k in turn.

b. If V is a terminal vertex in T_x, let the new area v^1 of V equal the area v of V in T.

Otherwise, V is adjacent to vertices $W_1, W_2, \ldots, W_m$ each of level $k+1$ with new areas $w_1^1, w_2^1, \ldots, w_m^1$.

Let the new area v^1 of V be the sum of v and $w_1^1, w_2^1, \ldots, w_m^1$.

c. Replace k by $k-1$.

d. If $k \neq 1$ go to a.

<u>Step 3</u>: Initial division of polygon.

a. Draw a line σ parallel to side α_2 across the given polygon F dividing it into two convex polygons, so that the one containing side α_2 has area x, the area of the root.

b. Label this polygon.

c. Take all the level one vertices, $Y_1, Y_2, \ldots, Y_\ell$ (with new areas $y_1^1, y_2^1, \ldots, y_\ell^1$). Divide the unlabelled polygon with lines parallel to α_1, into polygons with areas $y_1^1, y_2^1, \ldots, y_\ell^1$ and label them $Y_1, Y_2, \ldots, Y_\ell$ respectively.

<u>Step 4</u>: Complete division.

For each level k, starting with $k=1$.

a. Take each vertex W of level k. W is already placed in the floorplan. W is adjacent to vertices $V_1,\ldots,V_\ell$ in T_x each of level $k+1$.

b. The polygon labelled W has one wall, say υ parallel to side α_2 not on the boundary of F.
Erase the label W from this polygon.

c. Draw a line across this unlabelled polygon parallel to side α_2 dividing it into two convex polygons, so that the polygon containing wall υ has area w, the area of vertex W. Label this polygon W.

d. An unlabelled polygon is left. Divide this with lines parallel to α_1 into polygons with areas $\upsilon_1{}^1,\ldots,\upsilon_\ell{}^1$ and label them $V_1,\ldots,V_\ell$ respectively.

e. Repeat a to d above with k replaced by $k+1$ until all vertices in the tree are placed in F.

An example of this algorithm with areas omitted is shown in figure 6.1. Vertex A is chosen as the root.

<u>Theorem 6.1</u> Consider a parabolic polygon F with area $A(F)$. Let T be a tree with n vertices $A,B,\ldots,N$ representing the required internal adjacencies between n rooms. Further let each vertex in the tree have an associated area - the required area of the corresponding room, with the sum of the areas equalling $A(F)$. Then it is possible to divide the polygon into convex rooms labelled $A,B,\ldots,N$ forming a floorplan so that both the required adjacencies and areas are satisfied. That is, the weak dual of the floorplan has T as a spanning tree.

<u>Proof:</u> Use the algorithm outlined above.

In the initial division of the polygon, <u>Step 3c</u>, the unlabelled polygon has side σ adjoining sides ρ and τ with ρ,σ,τ appearing in sequence anticlockwise around its perimeter. Since side σ is parallel to edge α_2, the size of the angle between ρ and σ is at most equal to that

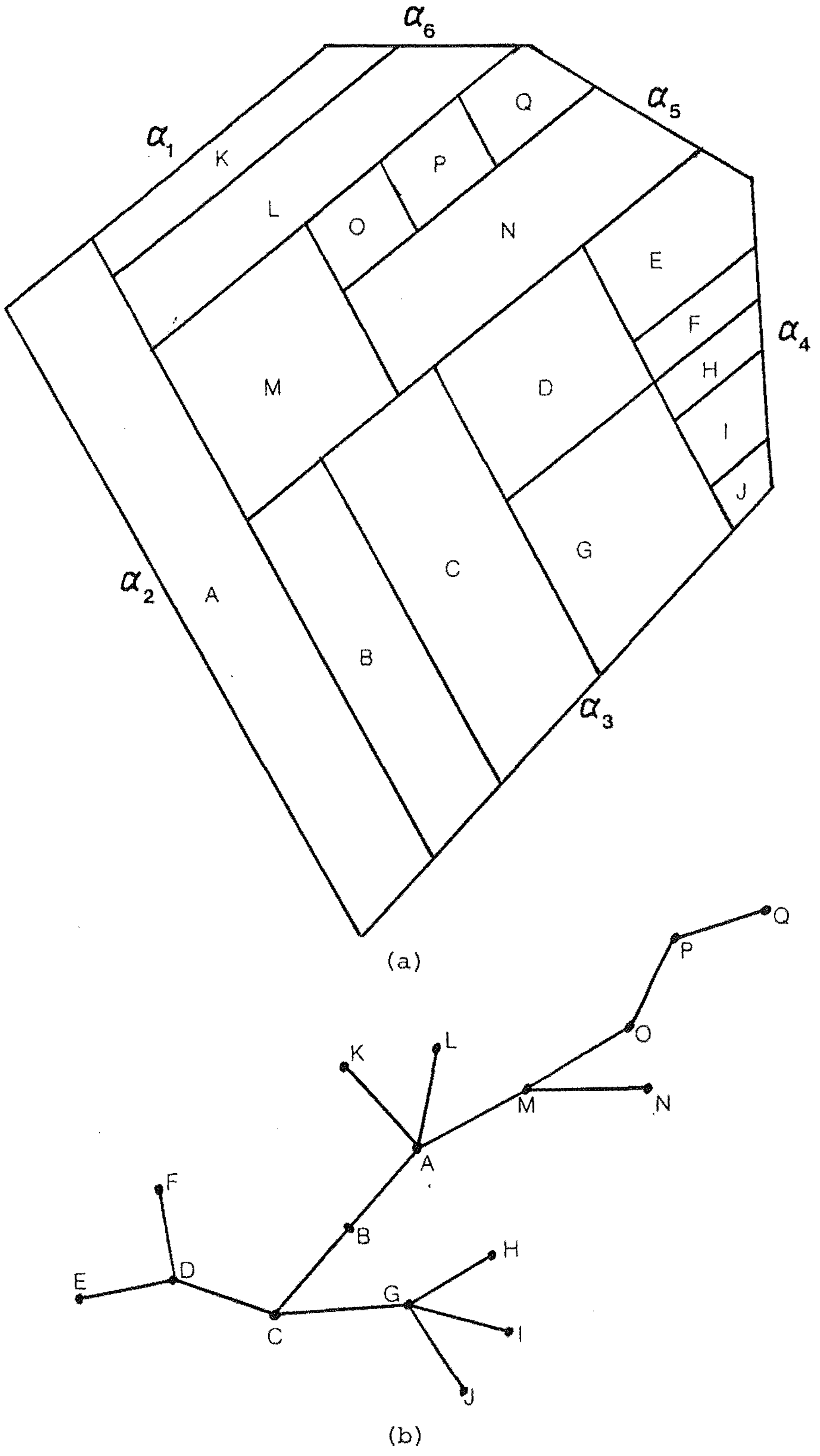


Figure 6.1 Division of a simple polygon (a) to suit adjacency requirements of T(b) using algorithm 6.1. A is the root. Areas are omitted.

between sides α_1 and α_2, and the size of the angle between σ and τ is at most equal to that between α_2 and α_3, so that the unlabelled polygon is also a parabolic polygon. Since the polygons labelled $Y_1, Y_2, \ldots, Y_k$ are formed by lines parallel to α_1, they will each have one wall in common with polygon X as required, and also be parabolic polygons. Similarly in Step 4d the polygons $V_1, \ldots, V_\ell$ will each be adjacent to V as required.

Hence using the above algorithm, at each step of the division of F both the area and adjacency requirements of the vertices in T are satisfied. Also each polygon is convex. The result follows. #

The following sections examine the tree adjacency problem for each of the rectangular, isometric and convex floorplan cases.

II THE RECTANGULAR TREE ADJACENCY PROBLEM

Theorem 6.2 The tree adjacency problem in the rectangular case is always solvable.

Proof: A rectangle is a parabolic polygon. Using algorithm 6.1, each of the polygons formed as a result of subdivision will be rectangular, as the rooms are formed by walls parallel to two adjacent walls in the given polygon. #

III THE ISOMETRIC TREE ADJACENCY PROBLEM

In an isometric floorplan all walls lie in one of three directions, the three directions being at 120° to each other.

If a given polygon F is to be the exterior boundary of an isometric floorplan then, provided it has fewer than six sides, it is a parabolic polygon. If it has six sides so that all interior angles are 120° and T is a tree as before, then consider the following algorithm to divide F into an isometric floorplan.

Algorithm 6.2 For an isometric floorplan with six sides.

Let F be the given polygon having sides $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_6$ in consecutive order around its plan boundary where α_2 is a side of maximum length, and $\alpha_3 \geq \alpha_1$. Reorientate the floorplan as in figure 6.2. Then drawing a line β across the polygon bisecting the angle between sides α_3 and α_4 divides the floorplan into two polygons so that the one containing side α_4 has area a_1. Adding another line γ across F bisecting the angle between sides α_1 and α_6 further divides F so that the polygon containing side α_2 has area a_2 and the polygon containing both lines β and γ has area a_3.

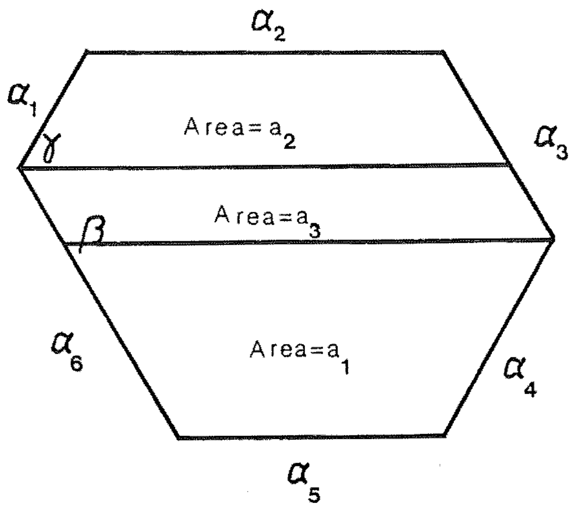


Figure 6.2 A six sided isometric polygon.

Step 1: Select root and assign levels.

a. Case 1 If any vertex X in the tree T has given area x at least equal to a_1 let X be the root of the tree T_x.

b. If not, then for any vertex W in the tree, the forest T-W consists of k subtrees where $k = \deg_T(W)$. The set of these subtrees can be partitioned into two sets S_1 and S_2 with the sum of the areas of the vertices in S_1 or S_2 equalling s_1, or

s_2 respectively. S_1 may be a null set in which case $s_1 = 0$. More than one partition may exist for every W.

<u>Case 2</u> If some vertex X has a partition S_1, S_2 such that $x \leq \alpha_3$, $s_1 \geq \alpha_1$ and $s_2 \geq \alpha_2$, then let X be the root of the tree T_x.

<u>Case 3</u> Otherwise for every W in T, select the partition S_1^W and S_2^W which minimizes $\alpha_1 - w - s_1$ subject to $w + s_1 \leq \alpha_1$. Then select the vertex X in T which minimizes $\alpha_1 - x - s_1^X$. Thus $x + s_1^X \geq w + s_1^W$ for any vertex W in T. Let X be the root of the tree T_x.

c. Assign levels as in algorithm 6.1

<u>Step 2</u> Assign a new area v^1 to each vertex V.

Assign these as in algorithm 6.1.

<u>Step 3</u> Division of polygon F.

a. <u>Case 1</u> If X was chosen in case 1 above, then draw a line σ parallel to side α_5 across F dividing it into two so that the polygon containing side α_5 has area x.

Use <u>Steps</u> 3c and 4 of algorithm 6.1 to complete the division but replace α_1 by α_3.

Figure 6.3 illustrates the initial division in this case.

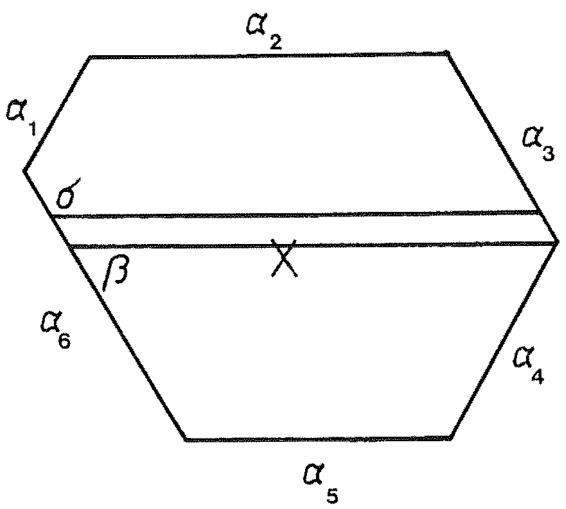


Figure 6.3 The initial division of polygon F for case 1 in algorithm 6.2.

b. <u>Case 2</u> If X was chosen in case 2 in <u>Step 1</u>, then draw 2 lines σ_1 and σ_2 parallel to α_5 across F dividing it into three polygons so that the one containing side α_5 and line σ_1 has area s_1, the one containing side α_2 and line σ_2 has area s_2 and the remaining one has area x. Label these polygons S_1, S_2 and X respectively.

Use <u>Steps</u> 3c and 4 of algorithm 6.1 to complete the division of S_1 restricting the vertices to those whose area is included in S_1 and replacing α_1 by α_6. Then use <u>Steps</u> 3c and 4 of algorithm 6.1 to complete the division of S_2 restricting the vertices those whose area is included in S_2 and replacing α_1 by α_3. Figure 6.4 illustrates the initial division in this case.

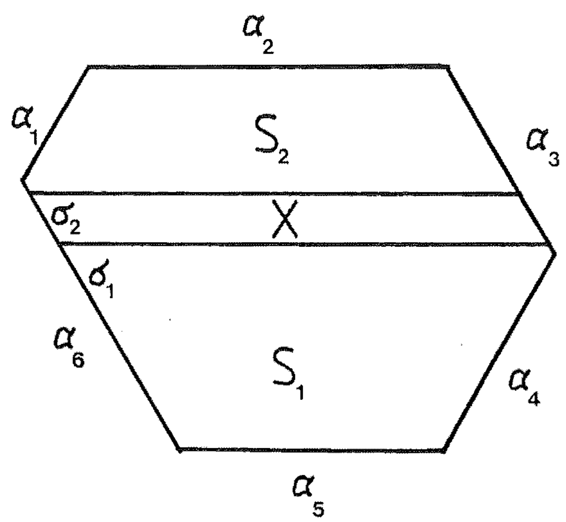


Figure 6.4 The initial division of polygon F for case 2 in algorithm 6.2.

c. Case 3 If X was chosen in case 3 in Step 1, then draw 2 lines σ_1 aand σ_2 parallel to side (or wall) α_5 across F dividing it into three polygons so that the one containing wall α_5 and line σ_1 has area s_1^X, the one containing α_2 and line σ_2 has area s_2^X and the remaining one has area x. Label these polygons S_1, S_2 and X respectively. This is shown in figure 6.5. Let ϵ be the distance between σ_2 and β measured along α_4. Then Step 1 chooses X and S_1^X to minimize ϵ subject to $\epsilon \geq 0$. If $S_1^X = 0$ the situation is as in figure 6.5(a); otherwise (b) applies.

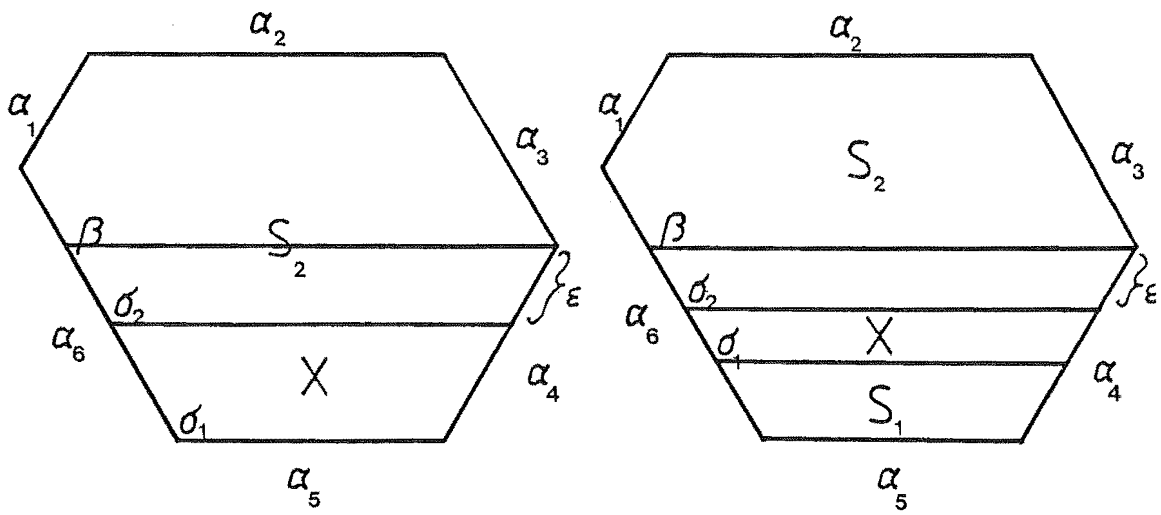


Figure 6.5 The initial division of isometric polygon F for case 3 in algorithm 6.3. The wall σ_2 lies parallel to and between α_s and β. In (a) $s_1^X = 0$.

(i) If $S_1^X \neq 0$, then divide the polygon labelled S_1, after removing its label, using Steps 3c and 4 of algorithm 6.1 restricting the vertices to those whose area is included in s_1^X and replacing α_1 by either α_4 or α_6.

(ii) The set S_2^X consists of ℓ elements which are subtrees of T-X.

(a) If $\ell = 1$, divide the polygon labelled S_2, after removing its label, using <u>Steps</u> 3c and 4 of algorithm 6.1 restricting the vertices to those whose area is included in s_2^X and replacing α_1 by α_3.

(b) If $\ell \geq 2$, let S_2^1 be the subtree in S_2^X for which the sum of the area of the vertices in it is the least. Further, let Y_1 be the vertex of S_2^1 which is adjacent to X in T. (Thus from <u>Step 2</u> above, y_1^1 equals the sum of the areas in S_2^1). Take all the level one vertices Y_1, Y_2, ..., Y_ℓ whose areas y_1, y_2, ..., y_ℓ are contained in s_2^X. Divide the polygon labelled S_2, after removing its label, with lines parallel to α_3 into polygons with areas y_1^1, y_2^1, ..., y_ℓ^1 so that the one containing side α_3 has area y_1^1. Label them Y_1, Y_2, ..., Y_ℓ respectively.

(α) If S_2^1 contains at least two vertices, then place a line parallel to α_3 across the polygon labelled Y_1, after removing its label, dividing it into two polygons so that the one not containing side α_3 has area y_1. Label this polygon Y_1. Y_1 is adjacent to vertices V_1, ..., V_ℓ in T_x each of level 2. Draw lines parallel to α_2 across the unlabelled polygon dividing it into polygons with areas v_1^1, ..., v_ℓ^1 and label them V_1, ..., V_ℓ respectively. If there are still vertices in S_2^1 which have not yet been placed in F, then use Step 4 of algorithm 6.1 beginning with $k = 2$, replacing α_1 by α_2, and α_2 by α_3 until they are all positioned.

(β) Divide the polygons labelled $Y_2, \ldots, Y_\ell$ using Step 4 of algorithm 6.1 restricting the vertices to those in the subtrees other than S_2^1 in the set S_2^X, and replacing α_1 by α_3.

Lemma 6.3 Let $ABCDEF$ be a hexagon with all its interior angles 120° and with AB a side of maximum length. Then the lines bisecting the angle ABC and FAB both meet DE, possibly at E or D respectively.

Proof: Consider figure 6.6. The line ℓ through B bisecting angle ABC passes through the hexagon and so must meet its boundary again. As ℓ is parallel to AF and CD it does not meet them, and as it meets AB and BC at B it does not meet them again. Hence ℓ meets either DE or EF. Suppose ℓ meets EF in G strictly between E and F. Then $BAFG$ is an isosceles trapezium and $AB = FG < EF$. But AB is a side of maximum length, so we have a contradiction. Hence ℓ meets DE possibly at E. Similarly the line through A bisecting angle FAB meets DE, possibly at D. #

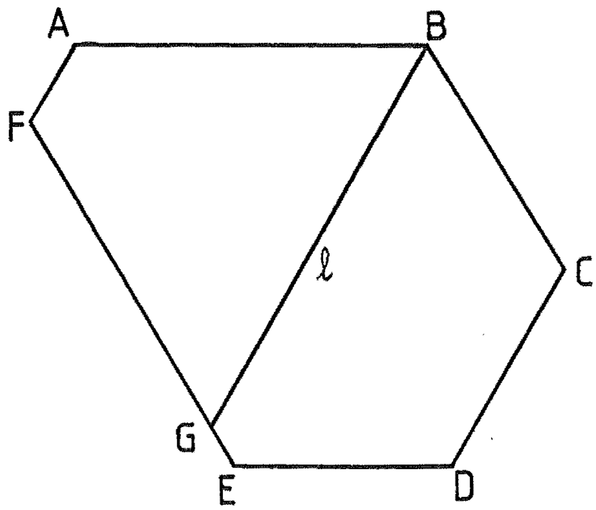


Figure 6.6 The hexagon for lemma 6.3.

Theorem 6.4 The tree adjacency problem in the isometric case is always solvable.

Proof: If the given polygon F has at most five sides then it must be a parabolic polygon. Hence by careful choice of the sides of F referred to as α_1,α_2 and α_3 in definition 6.1, the conditions of algorithm 6.1 are satisfied. In particular, if F has

(a) three sides, any choice of α_1,α_2 or α_3 will suffice;

(b) four sides, let α_1 and α_3 be two parallel sides of F;

(c) five sides, choose $\alpha_1,\alpha_2,\alpha_3$ so that α_1 and α_3 are parallel and α_2 is adjacent to α_1 and α_3.

Since the polygons using algorithm 6.1 are formed by lines parallel to either α_1 or α_2, each will be convex with walls lying parallel to one of three given directions. An isometric floorplan is thus formed. The area and adjacency requirements will be satisfied as was shown in theorem 6.1 earlier.

If the polygon has six sides, then use algorithm 6.2 to divide its interior.

Should case 1 apply, then when X is positioned in the polygon the unlabelled polygon is a parabolic polygon. This can be seen from figure 6.3 earlier. The use of algorithm 6.1 to complete the division ensures both area and adjacency requirements are satisfied, and that each room is convex.

Should case 2 apply, then the initial division of F as shown in figure 6.4 is such that the polygons labelled S_1 and S_2 are parabolic polygons. As before, using algorithm 6.1 to complete the division ensures both area and adjacency requirements are satisfied and that each room is convex.

Should case 3 apply, then the initial subdivision of F is as in figure 6.5. The choice of X and the partition S_1^X, S_2^X of the subtrees in

$T-X$ ensures that the distance ϵ is the least possible. In figure 6.5(b) S_1 is a parabolic polygon and so the algorithm ensures both area and adjacency requirements are satisfied, and each room is convex.

If S_2^X has only one element, S_2^1, then the algorithm places the vertex Y adjacent to X in T as in figure 6.7. B must be on AC and not on CD; otherwise ϵ has not been minimized as then Y would have been chosen as the root.

The unlabelled polygon in figure 6.7 is a parabolic polygon. The remainder of the algorithm ensures both area and adjacency requirements are met, and that each room is convex.

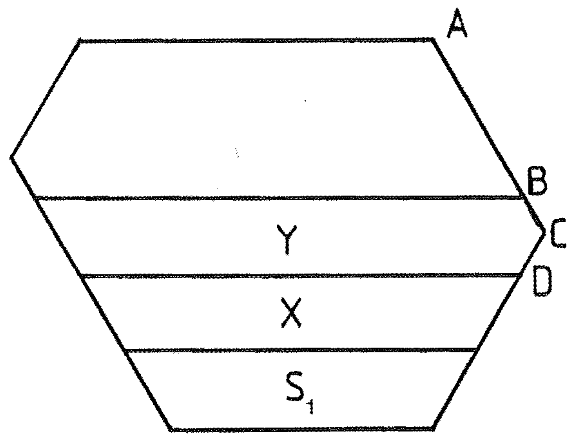


Figure 6.7 The initial division of S_2 for case 3 in algorithm 6.2 when S_2^X has only one element.

If the set S_2^X has at least two subtrees, then in Step 3(ii)(b) the polygon S_2 in figure 6.5(b) is divided by lines parallel to α_3, so that the subtree S_2^1 with the smallest area (y_1^1) contains edge α_3. This is shown in figure 6.8.

Each of Y_2, ..., Y_ℓ is a parabolic polygon and is further subdivided to satisfy adjacency and area requirements and to ensure each room is convex. Y_1 is also a parabolic polygon and if S_2^1 consists of more than one vertex then it is further subdivided in Step 3(ii)(b)(α) by drawing a line parallel to α_3 making the vertex Y_1 adjacent to X, and then using algorithm 6.1 to subdivide the remainder. This ensures adjacency and area requirements are met, and each room is convex.

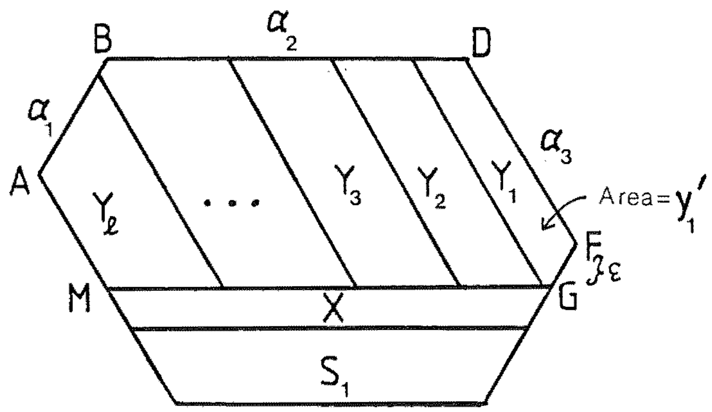


Figure 6.8 The initial division of S_2 for case 3 in algorithm 6.2 when S_2^X has at least two elements.

Two problems could occur.

The area of S_2^1 may be so small that Y_1 has no edge in common with X.

Consider figure 6.9.

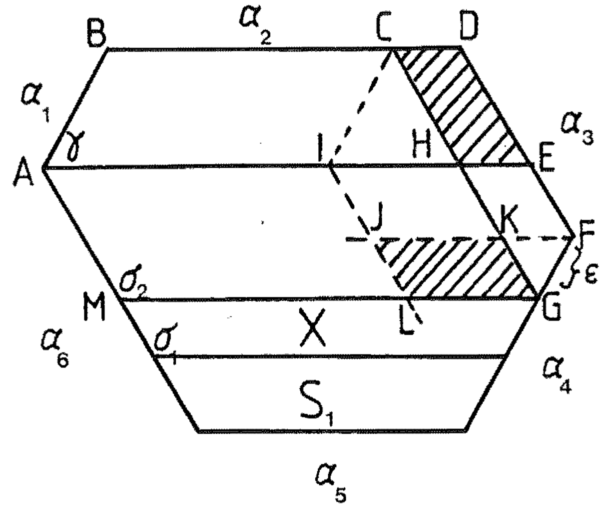


Figure 6.9 The division of F in theorem 6.4 showing area $(CDFG) <$ area $(AEFGM)$.

Here GC is the line through G parallel to α_3. GC cuts α_2 as CD = FG, and FG is part of α_4, while α_2 is the longest side of the floorplan F. Let CI be parallel to α_1 and IL parallel to α_3. Then shaded areas $CDEH$ amd $GLJK$ are equal. Hence area ($CDFG$) < area ($AEFGM$).

The subtree S_2^1 in S_2^X with the smallest area, y_1^1, must have area greater than area ($AEFGM$); otherwise by removing S_2^1 from the set S_2^X and adding it to set S_1^X either the minimality of ϵ is contradicted or $s_1^X + y_1^1 \geq \alpha_1$ and $s_1^X - y_1^1 \geq \alpha_2$ but then <u>case 2</u> applies (see <u>Step 1</u>). Thus y_1^1 > area ($CDFG$). Hence Y_1 in figure 6.8 has one wall in common with polygon X.

The other problem is that the area of S_2^1 is so large that Y_1 in figure 6.8 contains the point B and hence is no longer a parabolic polygon.

Draw in the two lines bisecting the angles between α_1 and α_2, and α_2 and α_3.

These two lines may intersect at a point below σ_2 as shown in figure 6.10(a). As $\alpha_3 \geq \alpha_1$, $AJ \geq \epsilon$. Thus area ($DFGH$) > area ($ABIJ$), and so also area ($BDFGI$) > area ($ABIJ$). Thus Y_1 in figure 6.8 cannot include point B for then S_2^1 has not got the smallest area of all the subtrees in S_2^X.

Otherwise they intersect at a point above σ_2 as shown in figure 6.10(b). From lemma 6.3 both L and K are on MN, and so $KL \leq MN \leq \alpha_2$. Again as $\alpha_3 \geq \alpha_1$, $AP \geq \epsilon$. Also $BJ = DJ = \alpha_2$ and $IJ = JH$. Thus area ($DFGHJ$) $\geq$ area ($ABJIP$). Also $IJ < KL \leq \alpha_2$. Thus area (BDJ) > area (JKL). Hence area ($BDFGH$) $\geq$ area ($ABHP$). Once again Y_1 in figure 6.8 cannot include point B for then S_2^1 has not got the smallest area of all the subtrees in S_2^X.

Thus, algorithm 6.2 divides any six sided isometric floorplan into an isometric floorplan satisfying both area and adjacency requirements.

The result of the theorem follows. #

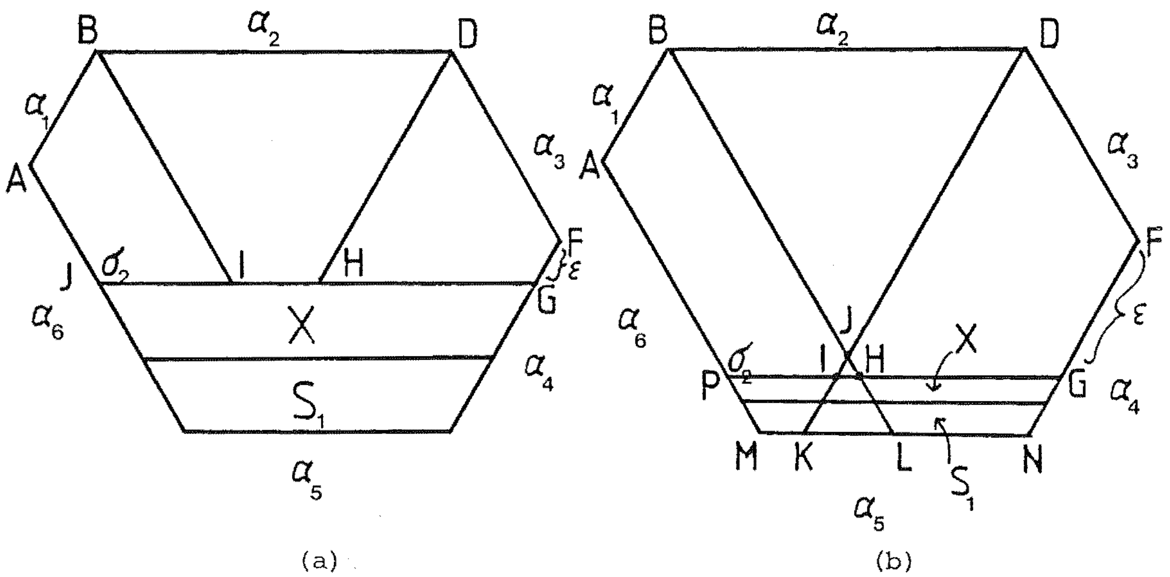


Figure 6.10 The division of F in theorem 6.4 to show polygon Y_1 cannot include point B. In (a) the bisectors of angles B and D meet below σ_2, while in (b) they intersect at J above σ_2.

IV THE CONVEX TREE ADJACENCY PROBLEM

Here the rooms and boundary of the floorplan are convex with walls lying in any direction. Consider the following algorithm.

Algorithm 6.3 For a convex floorplan.

Step 1: Select root and assign levels.

As for Step 1 in algorithm 6.1.

Step 2: Assigning new areas.

As for Step 2 in algorithm 6.1.

Step 3: Initial division of polygon.

As for Step 3 in algorithm 6.1 letting α_1 in (a) be any edge in the polygon, and in (c) replacing "lines parallel to α_1," by "lines which when extended beyond the polygon intersect at a common point, the midpoint of edge α_2".

Step 4: Complete division.

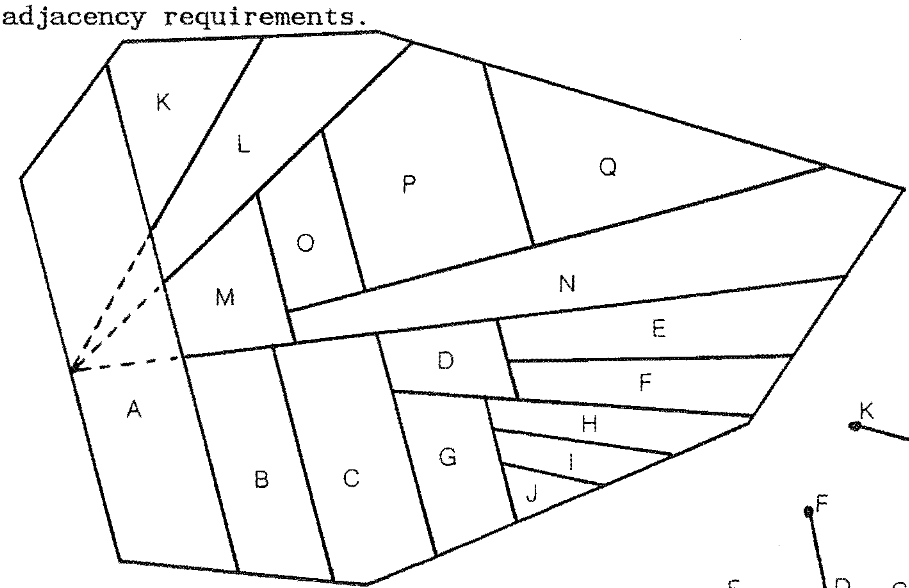
As for Step 4 in algorithm 6.1, making the same replacement of "lines parallel to α_1" as in Step 3 above.

Figure 6.11 is an illustration of this algorithm.

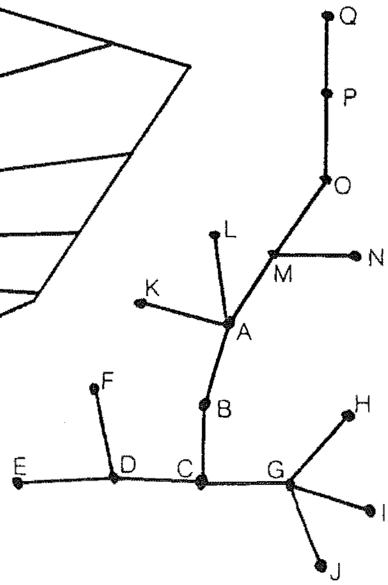
Theorem 6.5 The tree adjacency problem in the convex case is always solvable.

Proof: If the given polygon is a simple polygon, algorithm 6.1 ensures the problem is possible. Otherwise use algorithm 6.3. The method of dividing the polygon in Step 3 of algorithm 6.3 ensures each $Y_1, Y_2, \ldots, Y_k$ is adjacent to X and has the correct area. Since the remainder of the algorithm is similar to algorithm 6.1, the floorplan formed will satisfy both area and adjacency requirements given by T and be a convex floorplan. #

Note: Algorithm 6.3 could also be given to divide a parabolic polygon. In fact the three algorithms given in this chapter are only several of the many that exist to create floorplans satisfying certain area adjacency requirements.



(a)



(b)

Figure 6.11 Division of a convex polygon (a) using algorithm 6.3 to satisfy the adjacency requirements of T (b). A is the root. Areas have been omitted.

CHAPTER VII

TREE ADJACENCY PROBLEM WITH EXTERNAL ROOMS

In chapter VI it was shown that the tree adjacency problem for either a rectangular, isometric or convex floorplan is always possible. Chapter V showed certain maximal outerplanar graphs with associated areas for the vertices were not the weak duals of any dimensioned rectangular or isometric floorplan.

Now we investigate whether the tree adjacency problem is possible for proper rectangular and isometric floorplans. Such a floorplan will have a weak dual, G, which is maximal outerplanar, and the given tree will be a spanning subtree of G. Each room must be external. Theorems from chapter III concerning embeddings of trees in maximal outerplanar graphs are needed here.

I RECTANGULAR FLOORPLANS

Theorem 7.1 No tree T for which $\epsilon(T) > 2$, is a spanning tree of the weak dual of any proper rectangular floorplan.

Proof: A proper rectangular floorplan has a weak dual G which is maximal outerplanar with at most four vertices of degree 2. If T is a tree with its embedding index $\epsilon(T) = r$, then by corollary 3.24, any maximal outerplanar graph in which T is embeddable has at least $r + 2$ vertices of degree 2. Thus if T is embeddable in G, that is, T is a spanning tree of G, then $\epsilon(T) \leq 4-2 = 2$. The result follows. #

Corollary 7.2 The tree adjacency problem for a proper rectangular floorplan cannot always be solved.

A necessary condition for a tree T to be a spanning tree of the weak dual of a proper rectangular floorplan therefore is $\epsilon(T) \leq 2$. From theorems 4.5 and 4.7 in chapter IV we know a dimensioned proper rectangular floorplan can always be found having a given maximal outerplanar graph G as its weak dual, and satisfying any area requirements, if G has two or three vertices of degree 2. From this we have the following theorem:

<u>Theorem 7.3</u> Given a tree T with $\epsilon(T) \leq 1$, and associated areas for every vertex, a proper rectangular floorplan can always be found satisfying the adjacency and area conditions given by T.

<u>Proof:</u> If $\epsilon(T) \leq 1$, so that $\epsilon(T) = 0$ or 1, then by theorem 3.28 T can be embedded as a spanning tree in a maximal outerplanar graph G with exactly two or three vertices of degree 2. By theorems 4.5 and 4.7 a proper rectangular floorplan can be found satisfying the area and adjacency requirements given by G and hence also by T. #

<u>Theorem 7.4</u> A proper rectangular floorplan cannot always be found satisfying the adjacency and area conditions given by tree T if $\epsilon(T) = 2$.

<u>Proof:</u> If $\epsilon(T) = 2$, then by corollary 3.29, the minimum number of vertices of degree 2 of any maximal outerplanar graph G, in which T can be embedded is four. If G has exactly four vertices of degree 2, then by theorem 4.4, a proper rectangular floorplan cannot always be found to satisfy the adjacency and any area requirements given by G. If G has at least five vertices of degree 2, no proper rectangular floorplan can be found having G as its weak dual by theorem 4.1. The result of the theorem follows. #

Although a proper rectangular floorplan cannot always be found if $\epsilon(T) = 2$, it is always possible to find an exterior rectangular floorplan. That is, one (see definition 4.1) in which each room is external, and 4-joints and through rooms may occur. This is shown in the following theorem.

Theorem 7.5 Let T be a tree for which $\epsilon(T) = 2$, and every vertex has an associated area. Let the sum of the areas equal $A(F)$. Then any rectangle with area $A(F)$ can be divided to form an exterior rectangular floorplan, so that T is a spanning tree of the plan's weak dual, and each room has the required area.

Proof: There are two cases to consider:-

(a) exactly one vertex X has $\beta(X) = 4$, or

(b) there are two vertices X and Y in T with $\beta(X) = \beta(Y) = 3$.

Detailed algorithms to produce floorplans for both of these exist, case (a) being a simpler version of (b). From a redrawing of the tree the floorplan can be easily derived. A worked example of a case (b) tree is shown in figure 7.2.

First the tree T is redrawn. The subtree D of T induced by vertices with $\beta \geq 2$, and each vertex W where $\beta(W) = 1$ and W is adjacent in T to V with $\beta(V) \geq 2$, such that the subtree of $T-V$ containing W is branching, is drawn as in figure 7.1.
The terminal vertices in D have $\beta = 1$ in T. X and Y have $\beta = 3$ while all other vertices in D have $\beta = 2$ in T. X may be adjacent to W in which case $i = 0$. Similarly any of j,k,ℓ or m may be zero.

The remaining vertices of T are positioned in the regions marked (i), (ii) or (iii). Any vertex in a subtree of $T-Z$ rooted at a vertex which is not in D, but adjacent to Z in T, where Z lies on the path from W_1 to W_2, is placed in the region (i). Similarly, any vertex in a subtree of T-Z rooted at a vertex which is not in D but adjacent to Z in T, where Z lies on the path from W_3 to W_4, is placed in the region marked (iii). All other vertices are placed in region (ii).

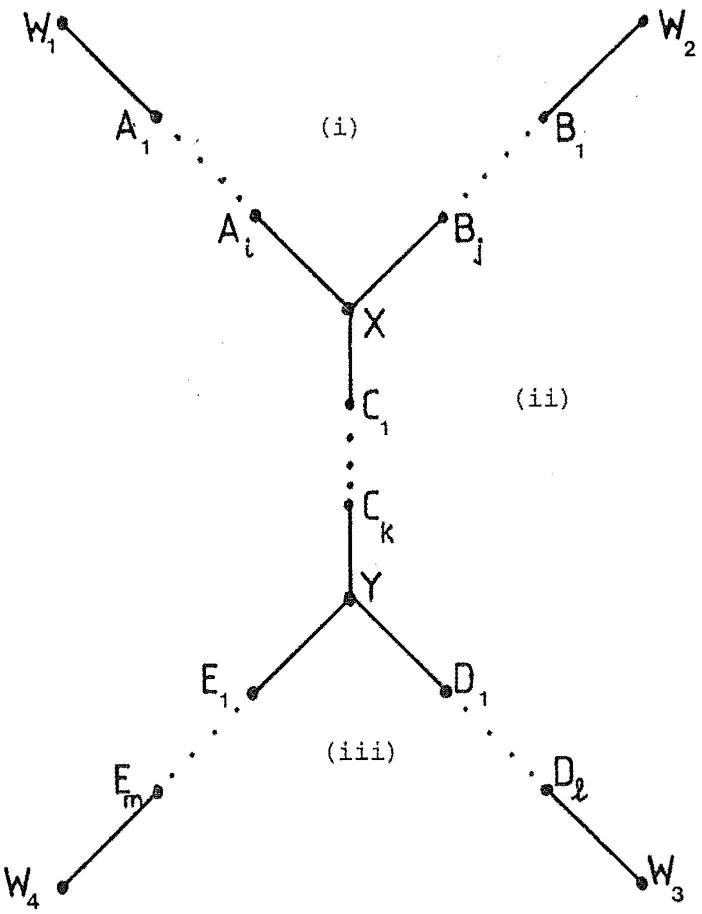


Figure 7.1 The subtree D of T.

From this redrawing of T, the floorplan can easily be derived. This is shown in figure 7.2. I and B' have branching index 3, while all other circled vertices have $\beta = 2$. The subtree D outlined above consists of all circled vertices and E, K, H' and Q'.

I and B' become through rooms in the plan. Vertices of D become adjacent to the west and east sides of the plan. The vertices of T not in D, positioned in regions (i), (ii) and (iii), are placed adjacent to the north, east and south sides respectively. The ordering of rooms

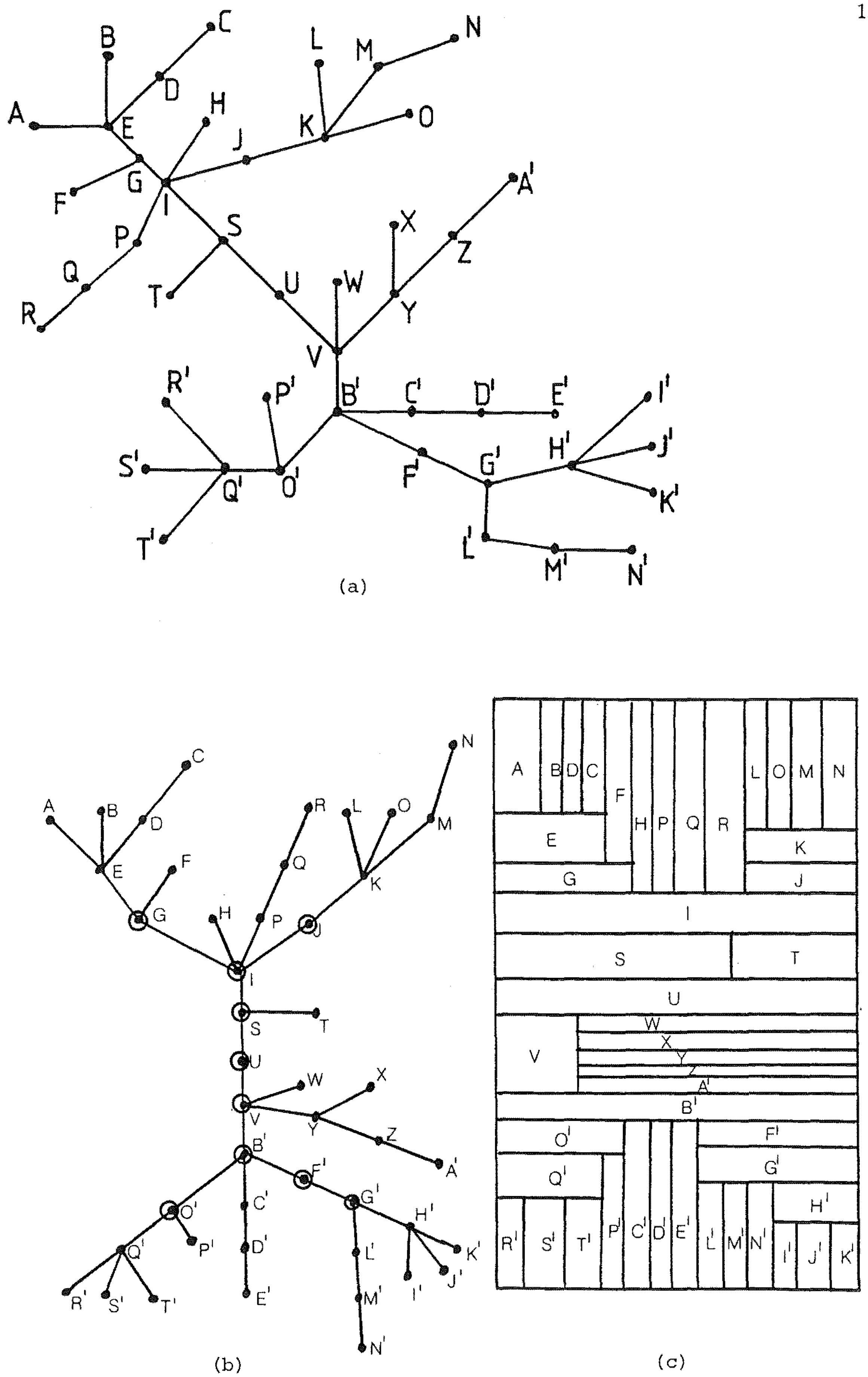


Figure 7.2   The exterior rectangular floorplan (c) suiting the adjacency requirements of *T* (a). *T* is redrawn in (b) according to theorem 7.5.

across or down the page corresponds to the positioning of vertices in the redrawn version of T.

A case (a) tree would correspond to a floorplan like figure 7.2(c) with rooms S to B' deleted. The result of the theorem follows. #

Remark: The algorithm used in theorem 7.4 could also be used to produce an exterior rectangular floorplan when $\epsilon(T) < 2$. If $\epsilon(T) = 1$, so that exactly one vertex X in T has $\beta(X) = 3$, then the floorplan produced would be like figure 7.2(c) with rooms S to B' and F' to N' deleted. If no vertex in T has branching index greater than 2, so that $\epsilon(T) = 0$, the floorplan would correspond to rooms A to N of figure 7.2(c), or to rooms H to N if no vertex in T has branching index of 2.

This avoids the need to embed T in a maximal outerplanar graph as in theorem 7.3, but is only one of the many algorithms that exist to produce an exterior rectangular floorplan satisfying the requirements of T.

II ISOMETRIC FLOORPLANS

Consider the tree T shown in figure 7.3 consisting of 101 vertices: one labelled 0 having degree 25, 25 labelled $D_1, D_2, D_3, \ldots, D_{25}$ having degree 4 and 75 labelled $A_1, B_1, C_1, A_2, B_2, C_2, \ldots, A_{25}, B_{25}, C_{25}$ having degree 1 with each D_i adjacent to A_i, B_i and C_i.

The only vertex with branching index greater than 2 is 0. In fact, $\beta(0) = 25$, as the forest T-0 consists of 25 trees, each induced by the vertices A_i, B_i, C_i and D_i of T. By corollary 3.29 earlier, any maximal outerplanar graph G with T as a spanning tree must have at least 25 vertices of degree 2. One of each A_i, B_i or C_i in T, for $i = 1$ to 25, corresponds to a vertex of degree 2 in G. It can be assumed without loss of generality that at least each A_i corresponds to a vertex of degree 2 in G.

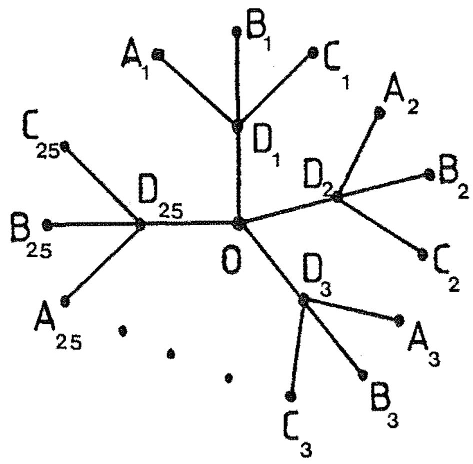


Figure 7.3 The tree for the counterexample.

Theorem 7.6 The tree adjacency problem for a proper isometric floorplan is not always solvable.

Proof: Let the given isometric polygon F be a regular hexagon, with each side of length c, so that its perimeter is 6c, and area $A(F)$ equals $\frac{3\sqrt{3}c^2}{2}$.

Consider the tree T of figure 7.3 with each A_i, B_i and C_i having area $\frac{1}{100}A(F)$ and each D_i and 0 having area $\frac{25}{2600}A(F)$, for $i = 1$ to 25, associated with it.

If the hexagon is to be divided into rooms A_i, B_i, C_i and D_i, for $i = 1$ to 25, and 0 to form a proper isometric floorplan having the areas and at least the adjacencies given by T satisfied, then the weak dual G of the floorplan must be maximal outerplanar with at least 25 vertices of degree 2. Without loss of generality it can be assumed that these vertices must include each A_i of the tree T.

In chapter V it was shown a vertex of degree 2 in the maximal outerplanar weak dual G of an isometric floorplan corresponds to either a corner room or a non-corner room which is an equilateral triangle. A 2-(or 3 or 4 or 5) corner room has at least one wall of length c in common with the plan boundary. A 1-corner room with area a must be either a parallelogram or trapezium with the minimum length of wall on the plan boundary being $3^{-1/4}.\sqrt{8a}$ or $3^{-1/4}.\sqrt{4a}$ respectively. The length of wall coincident with the plan boundary of a non corner room which is an equilateral triangle room, with area a, is $3^{-1/4}.\sqrt{4a}$. Each A_i has area $a = \frac{1}{100} A(F) = \frac{1}{200} \times 3\sqrt{3c^2}$, so $3^{-1/4}\sqrt{4a} = \frac{1}{10} \times \sqrt{6}\, c < c$.

Thus the minimum length of wall common with the plan boundary of any room corresponding to a vertex of degree 2 in G is $\frac{1}{10} \times \sqrt{6}\, c$.

The total perimeter of the plan boundary must therefore exceed $\frac{25}{10} \times \sqrt{6}\, c$. As $\sqrt{6} < 2.5$, the given hexagon F having perimeter 6c cannot be divided to form a proper isometric floorplan satisfying the requirements of T. #

This theorem can be extended. Consider the problem of finding a proper isometric floorplan to suit the area and adjacency requirements of T as given in theorem 7.4, when any isometric polygon F having area $A(F)$ can be chosen. Then since F has at most six sides and corners, at least one side of F must have at least four non corner rooms, corresponding to vertices of degree 2 in the weak dual of the plan, along it. Thus four rooms say A_1, A_2, A_3 and A_4 which are equilateral triangles appear along this side.

There are various different ways in which the rooms can be arranged along this side corresponding to the order of the vertices in the bounding circuit of the floorplan's weak dual. One such order is

$$D_1, A_1, B_1, C_1, C_2, B_2, A_2, D_2, C_3, B_3, A_3, D_3, D_4, A_4, B_4, C_4$$

as shown by the dotted line in figure 7.4.

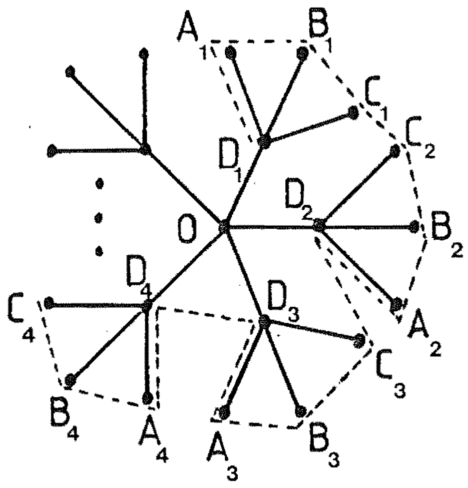


Figure 7.4 Part of the bounding circuit, shown dotted, of one embedding of the tree in figure 7.3 in a maximal outerplanar graph.

Thus we have the situation shown in figure 7.5 for the rooms along this side, as each D_i is adjacent to B_i and C_i and is convex. Further room O is to be adjacent to D_1, D_2, D_3 and D_4. It can be shown that if room O is convex then it must have a wall parallel to the side along which D_1 to D_4 lies.

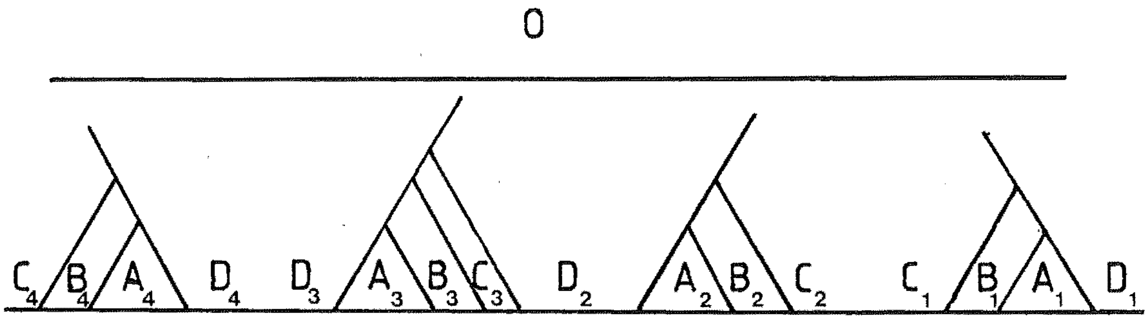


Figure 7.5 The positioning of rooms and walls along one side of an isometric polygon for the situation in figure 7.4.

Consider now the rooms in figure 7.6. As D_3 is adjacent to C_3, and D_2 to C_2, the joints α and β between rooms O, D_3,C_3 and O,D_2,C_2 are as shown. The wall between rooms C_3 and D_2 has not yet been positioned. As both C_3 and D_2 are convex and external, this wall can lie in either of two positions shown in the figure. In (a), the wall is parallel to and lies between the two dotted lines. Here the area of D_2 must exceed a_2, the area of A_2, which is a contradiction. Similarly, in (b) where the wall is parallel to and lies between the dotted line and the wall section from β to γ, the area of C_3 must exceed a_3, the area of A_3. This also is a contradiction. Thus this order of the rooms along the side of the floorplan is not feasible for a proper isometric floorplan satisfying both area and adjacency requirements given by T.

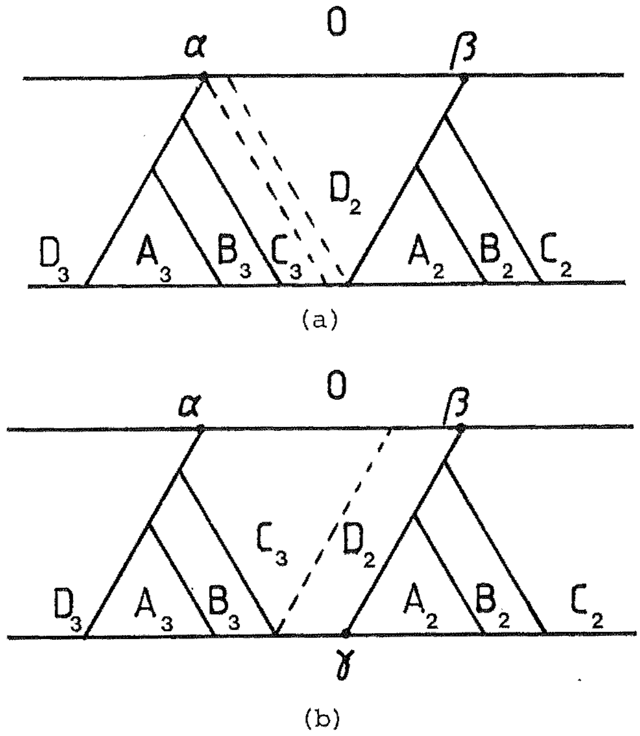


Figure 7.6 The two positions for the wall between rooms C_3 and D_2.

It can be shown by a similar argument that no other order of the vertices will give a feasible solution. This leads to the following theorem:

Theorem 7.7 No proper isometric floorplan F with area $A(F)$ can be formed to suit the internal adjacencies given by tree T in figure 7.3, and the area conditions of theorem 7.5. That is, each A_i, B_i and C_i has area $\frac{1}{100}A(F)$, and each D_i and O has area $\frac{25}{2600}A(F)$, for $i = 1$ to 25.

III CONVEX FLOORPLANS

This chapter has investigated the existence of proper rectangular and proper isometric floorplans that satisfy the area and adjacency requirements of a given tree. A proper convex floorplan can always be found as shown in the following theorem:

Theorem 7.8 A proper convex floorplan can always be found satisfying the area and adjacency requirements of a given tree T.

Proof: Embed T in a maximal outerplanar graph, G. By theorem 5.5, a proper convex floorplan can always be found satisfying the area and adjacency requirements of G, and hence also of T. #

CHAPTER VIII

SUMMARY, CONCLUSIONS AND FURTHER RESEARCH

However floorplans are designed, the choice is limited by what is geometrically and topologically possible.

Only a finite number of arrangements exist for the layout of rooms on a single floor with specified adjacencies between them, regardless of their shape or size. Limits on the geometry of the plan, and dimensional constraints reduce this variety of arrangement further.

This thesis has shown the nature of some of these limits by providing theorems for the existence of floorplans with given area and adjacency constraints.

Three types of floorplans - rectangular, isometric and convex were considered.

Proper floorplans have weak duals which are maximal outerplanar. We saw in chapter IV that weak duals of proper rectangular floorplans have at most four vertices of degree 2. Proper isometric or convex floorplans have no such restriction.

The design problem was then studied. Given a maximal outerplanar graph with at most four vertices of degree 2 and areas for each vertex or room, we investigated whether a proper rectangular floorplan could be found having the given graph as its weak dual and satisfying the area requirements.

This involved colouring the graph and then drawing the corresponding nonisomorphic dimensionless floorplans. These methods had been used earlier, as described in chapter II. However we were able, by exploiting the existence of a fault line in a proper rectangular floorplan, to draw the floorplan directly from the coloured graph. Each floorplan so formed had area conditions that needed to be satisfied if the plan was to be dimensioned to suit particular areas for the rooms.

By considering these plans and conditions we were able to show that a proper rectangular floorplan could always be found satisfying both area and adjacency requirements given by a weak dual G that was maximal outerplanar with two or three vertices of degree 2. However a solution could not be guaranteed if G had four vertices of degree 2.

We then showed that allowing through rooms in rectangular floorplans was often more restrictive.

Chapter V showed that proper isometric floorplans could not, while proper convex floorplans could, always exist satisfying area conditions and having any maximal outerplanar graph for its weak dual.

In chapter VI we showed that if the given adjacencies and areas could be represented in a tree, then any of the three types of floorplans could always be found satisfying all the constraints.

The tree in this case would be a spanning tree of the floorplan's weak dual. This led to an interesting question in graph theory - what restrictions exist on the embedding of a tree in a maximal outerplanar graph? This was answered in chapter III. The branching index was defined and its properties examined. From this, another index, ϵ, was used to give the minimum number of vertices of degree 2 of any maximal outerplanar graph in which a given tree can be embedded. Also a detailed algorithm and proof showed how this embedding could be achieved.

Chapter VII used these results. Here the given adjacencies and areas were represented by a tree. A floorplan was to be found in which each room was external and the given constraints were satisfied.

It was shown that a proper rectangular floorplan could always be found provided the embedding index of the given tree was 0 or 1. If this index was 2, then the only maximal outerplanar graphs in which the given tree could be embedded may be those for which no plan satisfying both adjacency and area conditions were possible. However it was shown an exterior rectangular floorplan in this case was always possible. If the index was greater than two, no rectangular floorplan was possible.

Under the same area and adjacency constraints, a proper isometric floorplan could not always be guaranteed. A proper convex floorplan, however, was always possible.

Thus three different types of adjacency constraints have been considered. Existence theorems for each type of floorplan with these adjacency constraints, as well as area constraints are now known. The results are summarized in Tables 8.1 and 8.2. Here "yes" denotes existence is always possible, "no", existence is never possible, while "sometimes" denotes existence cannot be guaranteed.

Table 8.1 Existence of floorplans to suit area and adjacency requirements given by tree *T*

Floorplan	$\epsilon(T)$	Existence
Rectangular	any	yes
Exterior Rectangular	≤ 2	yes
Proper Rectangular	0 or 1	yes
	2	sometimes
	> 2	no
Isometric	any	yes
Proper isometric	any	sometimes
Convex	any	yes
Proper convex	any	yes

Table 8.2 Existence of proper floorplans to suit area and adjacency requirements of G, a maximal outerplanar graph.

Floorplan	Vertices with degree 2	Existence
Proper rectangular	2 or 3	yes
	4	sometimes
	>4	no
Proper isometric	any	sometimes
Proper convex	any	yes

Under the conditions considered proper rectangular and isometric floorplans cannot always be found while proper convex plans are always possible. Moreover rectangular floorplans can never be found in some cases.

This thesis has only considered certain conditions and floorplans. Outerplanar graphs were studied in connection with areas. Adjacency graphs which are not outerplanar have at least one internal room. If this room or rooms have a large percentage of the total area of the plan, so that a large area is to be enclosed by a much smaller one, non-existence of a suitable floorplan is likely. In practice, many buildings, particularly domestic, require each room to be external.

Rectangular floorplans have taken up a large part of this thesis. This may appear somewhat restrictive. However, as was mentioned in chapter II, most buildings in practice are confined to a rectangular discipline. That is, all walls are parallel to one or two directions. A L-shaped room, for instance, can be represented by a pair of adjacent rectangles. Similarly other complex shapes can be broken down into rectangular pieces. Non-rectangular plan boundaries can be represented by the addition of dummy rooms adjacent to the exterior.

Not every maximal planar graph can be the weak dual of a rectangular floorplan. However the most recent work by Giffin (1986) and implemented by Keenan (1986) has shown that a dimensioned floorplan having a rectangular plan boundary and consisting of rooms either rectangular, *L* or *T*-shaped, can always be found having given areas and a given maximal planar graph which can be formed by the deltahedron method (Foulds and Robinson (1976)) as its weak dual.

This work is similar to this thesis. Clearly there is still much to be done in this area. We end with several comments and suggestions for future research on problems considered outside the scope of this thesis.

Existence theorems for rectangular floorplans with weak duals other than outerplanar can be considered. Often large rectangular floorplans can be divided by fault lines into smaller ones. Also a rectangular floorplan may have other rectangular floorplans enclosed within it. Further the conditions that the plan boundary be rectangular can be relaxed to include *L*, *U*, *T* or other related shapes. These areas require further study.

Other types of floorplans can be used, although choice should be limited to those that are reasonable in practice.

Recall that in chapter II we noted that the access graphs of most domestic dwellings and small buildings are trees. Chapter IV proved the existence of a rectangular floorplan having a given tree as a subgraph of its weak dual and satisfying area requirements.

A promising area of research in which this knowledge could be used is facilities layout. Here a plane region, usually a rectangle is given, as well as a number of rooms (or facilities) and an adjacency rating of the desirability of having each pair of rooms adjacent. The problem is to construct a floorplan so that all area requirements are satisfied, and the objective function - the sum of ratings of all adjacent rooms - is maximised. The exterior of the floorplan is often considered a facility.

One approach (Foulds and Robinson (1976), Giffin (1984), Hammouche and Webster (1985)), has two phases. First, a maximal planar graph representing the adjacencies is constructed, generally using some sort of heuristic. Next the floorplan having these adjacencies and the right areas is constructed. Some progress has been made in this direction. However, as mentioned above, if the plan boundary is rectangular, then the rooms cannot always be rectangular. Often, undesirable room shapes are created, long and very narrow for example, which are not of much practical use.

Other approaches, like ALDEP (Seehof and Evans (1967)) and CORELAP (Lee and Moore(1967)) build up the floorplan in one operation. Once again rooms of undesirable shapes may be formed.

Adjacencies are often based on a six point scale, *A-E-I-O-U-X*, (Muther (1973)). As adjacencies between rooms in most dwellings are usually to allow access, most of the ratings are *I*, *O* or *U*.

Thus the questions arises of whether it is possible to construct good rectangular floorplans, that is with rooms of reasonable proportion, and if so, the extent of the cost in terms of the objective function.

Rinsma and Robinson (1986) outlined a procedure that could be adopted. Their approach was to construct a high scoring tree of adjacencies, having as many *A*'s and *E*'s as possible and avoiding all *X*'s. The second phase would form a rectangular floorplan with this tree as a spanning tree of its weak dual, by an algorithm that picked up other worthwhile adjacencies and gave good room shape. A sketch of such an algorithm was given. Early results showed room shapes were superior compared to previous methods with a negligible cost in terms of the adjacency score. In fact sometimes the earlier methods were outscored.

Further research is necessary to create a testable algorithm.

A graph theoretic question left unanswered is an upper bound on the number of vertices of degree 2 in any maximal outerplanar graph in which a given tree can be embedded. This is likely to be related to the embedding index of the tree. Also the restrictions on the embedding of trees in graphs other than maximal outerplanar may be investigated.

It is envisaged that the ideas and results presented in this thesis could be used in answering these questions.

REFERENCES

AHO, A.V., HOPCROFT, J.E. and ULLMAN, J.D. (1974) *The Design and Analysis of Computer Algorithms* (Addison-Wesley, Reading, Mass.) pp 176-179.

BAYBARS, I. (1982) "The generation of floorplans with circulation spaces" *Environment and Planning B* 9, 445-456.

BAYBARS, I. and EASTMAN, C.M. (1980) "Enumerating architectural arrangements by generating their underlying graphs" *Environment and Planning B* 7, 289-310.

BEINEKE, L.W. and PIPPERT, R.E. (1972) "A census of ball and disk dissections" in Proceedings of a Conference on Graph Theory and Applications at Western Michigan University, May 10-13, 1972, published as "*Lecture Notes in Mathematics*" 303, (Springer-Verlag, Berlin) pp 25-40.

BEMIS, A.F. (1936) *The Evolving House* (Technology Press, MIT, Cambridge, Mass.).

BLOCH, C.J. (1976) "On the set and number of minimal gratings for rectangular dissections" *Environment and Planning B* 3, 71-74.

BLOCH, C.J. (1979a) *A Formal Catalogue of Small Rectangular Plans : Generation, Enumeration and Classification*, Ph.D. thesis, School of Architecture, University of Cambridge, Cambridge, England.

BLOCH, C.J. (1979b) "Catalogue of small rectangular plans" *Environment and Planning B* 6, 155-190.

BLOCH, C.J. and KRISHNAMURTI, R. (1978) "The counting of rectangular dissections" *Environment and Planning B* 5, 207-214.

BROOKS, R.L., SMITH, C.A.B., STONE, A.H. and TUTTE, W.T. (1940) "The dissection of rectangles into squares" *Duke Mathematical Journal* 7, 312-340.

COLBOURN, C.J. and BOOTH, K.S. (1981) "Linear time automorphism algorithms for trees, intevnal graphs, and planar graphs" *SIAM J. Comput.* 10, No.1, 203-225.

COMBES, L. (1976) "Packing rectangles into rectangular arrangements" *Environment and Planning B* 3, 3-32.

COUSIN, J. (1970) "Topological organization of architectural space" *Architectural Design* 40, 491-493.

EARL, C.F. (1977) "A note on the generation of rectangular dissections" *Environment and Planning B* 4, 241-246.

EARL, C.F. (1978) "Joints in two and three-dimensional rectangular dissections" *Environment and Planning B* 5, 179-188.

EARL, C.F. (1980) "Rectangular shapes" *Environment and Planning B* 7, 311-342.

EARL, C.F. (1981) "Enumerating architectural arrangements : comment on a recent paper by Baybars and Eastman" *Environment and Planning B* 8, 115-118.

EARL, C.F. and MARCH, L.J. (1979) "Architectural applications of graph theory" in *Applications of Graph Theory* Eds R.J. Wilson and L.W. Beineke, (Academic Press, London) pp 327-355.

FLEMMING, U. (1978) "Wall representations of rectangular dissections and their use in automated space allocation" *Environment and Planning B* 5, 215-232.

FLEMMING, U. (1980) "Wall representations of rectangular dissections : additional results" *Environment and Planning B* 7, 247-251.

FOULDS, L.R. and ROBINSON, D.F. (1978) "Graph theoretic heuristics for the plant layout problem" *Int. J. Prod. Res.* 16, No.1, 27-37.

FRIEDMAN, Y. (1975) *Towards a Scientific Architecture* translated by C. Lang (MIT Press, Cambridge, Mass.).

GALLE, P. (1981) "An Algorithm for Exhaustive Generation of Building Floor Plans" *Comm. ACM* 24, No.12, 813-825.

GALLE, P. (1986) "Abstraction as a tool of automated floor-plan design" *Environment and Planning B* 13, 21-46.

GERO, J.S. (1977) "Note on "Synthesis and optimization of small rectangular floor plans" of Mitchell, Steadman and Liggett" *Environment and Planning B* 4, 81-88.

GIFFIN, J.W. (1984) *Graph Theoretic Techniques for Facilities Layout*, Ph.D. Thesis, University of Canterbury, Christchurch, New Zealand.

GIFFIN, J.W. (1986) Personal communication.

GILLEARD, J. (1978) "LAYOUT - a hierarchical computer model for the production of architectural floor plans" *Environment and Planning B* 5, 233-241.

GIPS, J. (1975) *Shape Grammars and their Uses* (Birkhäuser, Basel).

GRASON, J. (1970) *Methods for the Computer-implemented Solution of a Class of 'Floor plan' Design Problems*, Ph.D. Thesis, Department of Electrical Engineering, Carnegie-Mellon University, Pittsburgh, PA.

GUTIÉRREZ, F. (1979) "Demonstration of Combe's formulae by use of the theory of graphs" *Environment and Planning B* 6, 301-304.

HAMMOUCHE, A. and WEBSTER, D.B. (1985) "Evaluation of an application of graph theory to the layout problem" *Int. J. Prod. Res.* 23, No.5, 987-1000.

HARARY, F. (1969) *Graph Theory* (Addison-Wesley, Reading, Mass.).

HASHIMSHONY, R., SHAVIV, E. and WACHMAN, A. (1980) "Transforming an Adjacency Matrix into a Planar Graph" *Building and Environment* 15, 205-217.

HILLIER, B. and HANSON, J. (1984) *The Social Logic of Space* (Cambridge University Press, Cambridge, England).

KEENAN, D.W. (1986) *Block Plan Construction from a Deltahedron Based Adjacency Graph*, M.Sc. Thesis, University of Arizona, Tucson, Arizona.

KORF, R.E. (1977) "A shape independent theory of space allocation" *Environment and Planning B* 4, 37-50.

KREJCIRIK, M. (1969) "Computer-aided plant layout" *Computer-Aided Design* 2, 7-19.

KRISHNAMURTI, R. (1979) "3 rectangulations : an algorithm to generate box packings" *Environment and Planning B* 6, 331-352.

KRISHNAMURTI, R. and ROE, P.H.O'N. (1978) "Algorithmic aspects of plan generation and enumeration" *Environment and Planning B* 5, 157-178.

KRISHNAMURTI, R. and ROE, P.H.O'N. (1979) "On the generation and enumeration of tessellation designs" *Environment and Planning B* 6, 191-260.

KRÜGER, M.J.T. (1979) "An approach to built-form connectivity at an urban scale : system description and its representation: *Environment and Planning B* 6, 67-88.

LEE, R.C. and MOORE, J.M. (1967) "CORELAP - COmputerized RElationship LAyout Planning" *Journal of Industrial Engineering* 18, 195-200.

LEVIN, P.H. (1964) "Use of graphs to decide the optimum layout of buildings" *Architect's Journal* 140, 809-815.

LUNNON, W.F. (1972) "Counting hexagonal and triangular polyominoes" in *Graph Theory and Computing* Ed. R. Read (Academic Press, New York) pp 87-100.

LYNES, J.A. (1977) "Windows and floor plans" *Environment and Planning B* 4, 51-56.

MARCH, L. and EARL, C.F. (1977) "On counting architectural plans" *Environment and Planning B* 4, 57-80.

MARCH, L. and MATELA, R. (1974) "The animals of architecture : some census results on n-omino populations for n=6,7,8" *Environment and Planning B* 1, 193-216.

MARCH, L. and STEADMAN, J.P. (1971) *The Geometry of Environment* (RIBA Publications, London).

MATELA, R. and O'HARE, E. (1976) "Graph-theoretic aspects of polyominoes and related spatial structures" *Environment and Planning B* 3, 79-110.

MITCHELL, S.L. (1979) "Linear algorithms to recognize outerplanar and maximal outerplanar graphs" *Information Processing Letters* 9, 229-232.

MITCHELL, W.J., STEADMAN, J.P. and LIGGETT, R.S. (1976) "Synthesis and optimization of small rectangular floor plans" *Environment and Planning B* 3, 37-70.

MUTHER, R. (1973) *Systematic Layout Planning* 2nd ed. (Cahners, Boston).

NEBESKY, L. (1976) "A generalization of hamiltonian cycles for trees" *Czechoslovak Mathematical Journal* 26 (101), 596-603.

NEWMAN, M.H.A. (1964) *Elements of the Topology of Plane Sets of Points* (Cambridge University Press, Cambridge).

OTTEN, R.H.J.M. (1982a) "Automatic floorplan design" in *Proceedings, 19th Design Automation Conference*, pp 261-267.

OTTEN, R.H.J.M. (1982b) "Layout structures" in *Proceedings, IEEE Large Scale Systems Symposium*, 1st IEEE International at Virginia Beach, U.S.A., 11-13 Oct, 1982 (New York) pp 349-353.

PROSKUROWSKI, A. (1979) "Minimum Dominating Cycles in 2-Trees" *International Journal of Computer and Information Sciences* 8, 405-417.

RINSMA, I. and ROBINSON, D.F. (1986) "Floorplans with better room shape" *New Zealand Operational Research* 14, No.2, 179-181.

ROBINSON, D.F. and JANJIC, I. (1985) "The constructability of floorplans with given outerplanar adjacency graph and room areas" *Ars Combinatoria* 20-B, 133-142.

ROTH, J., HASHIMSHONY, R. and WACHMAN, A. (1982) "Turning a Graph into a Rectangular Floor Plan" *Building and Environment* 17, 163-173.

SAUDA, E.J. (1975) *Computer Program for the Generation of Dwelling Unit Floor Plans*, MArch thesis, School of Architecture and Urban Planning, University of California, Los Angeles, CA.

SEEHOF, J.M. and EVANS, W.D. (1967) "Automated Layout Design Program" *Journal of Industrial Engineering* 18, 690-695.

SEPPÄNEN, J. and MOORE, J.M. (1970) "Facilities Planning with Graph Theory" *Management Science* 17B, 242-253.

STEADMAN, J.P. (1973) "Graph theoretic representation of architectural arrangement" *Architectural Research and Teaching* 2, 161-172; reprinted 1976 in *The Architecture of Form* Ed. L.J. March (Cambridge University Press, Cambridge) pp 94-115.

STEADMAN, J.P. (1983) *Architectural Morphology* (Pion, London).

STINY, G. (1975) *Pictorial and Formal Aspects of Shape and Shape Grammars* (Birkhäuser, Basel).

STINY, G. (1979) "Letter to the editor : Dissections dissected" *Environment and Planning B* 6, 469-470.

STINY, G. and MITCHELL, W.J. (1978) "Counting Palladian plans" *Environment and Planning B* 5, 189-198.

STOCKMEYER, L. (1983) "Optimal Orientations of Cells in Slicing Floor plan designs" *Information and Control* 57, 91-101.

SYSLO, M.M. (1979) "Characterizations of outerplanar graphs" *Discrete Mathematics* 26, 47-53.

SYSLO, M.M. (1982) "Graphs related to exterior rectangular dissections" *Congressus Numerantium* 36, 27-45.

TEAGUE, L.C. (1970) "Network models of configurations of rectangular parallelpipeds" in *Emerging Methods in Environmental Design and Planning* Ed. G.T. Moore (MIT Press, Cambridge, Mass.) pp 162-169.

VELEZ-JAHN, G. (1971) "Rectangular meshes : their uses and control in computer-produced architectural schemes" *Proceedings of the ACM 1971 Annual Conference* (Association for Computing Machinery, New York) pp 745-75.

ADDENDUM

ORTHOGONAL FLOORPLANS

This thesis has considered floorplans in which all rooms are convex and have walls parallel to one of several specified directions. In particular, in the rectangular case as all walls are adjacent to the plan boundary each room is a rectangle. An *orthogonal floorplan* has a rectangular plan boundary with the walls of each room parallel to the sides of the plan boundary. Non-convex rooms are permitted. This addendum investigates the existence of orthogonal floorplans with the three types of adjacency and area conditions considered in the main body of the thesis.

I. MAXIMAL OUTERPLANAR ADJACENCY GRAPHS

Theorem 4.4 stated that existence of a rectangular floorplan having a maximal outerplanar graph G as its weak dual and satisfying any area requirements for the rooms cannot be guaranteed if G has four vertices of degree 2. However an orthogonal floorplan may be possible in this case. In fact we have the stronger condition given in the following theorem.

<u>Theorem 9.1</u> An orthogonal floorplan in which each room is a rectangle or L-shaped can always be found having any given maximal outerplanar graph G as its weak dual and satisfying any room area requirements.

<u>Proof</u>: The constructive nature of the graph is used to assign levels to the vertices as described in theorem 3.11. That is, an initial triangle has vertices with levels 1, 2 and 3. Each vertex with level i, where $i>3$, is joined to two vertices with levels j and k where $i>j>k$. Then the

vertex with level i is the *successor* of the vertex with level j, while the vertex with level j is the *precursor* of that with level i.

If a vertex I with level i, joined to two vertices with levels j and k where $i>j>k$, has a successor L with level l, then L must also be adjacent to M with level m where $m<i$. Since G is maximal outerplanar, I,L,M are the vertices of a triangle in G and so I is adjacent to M. This implies either $m=j$ or $m=k$. Thus each vertex has at most two successors. The vertex with level 2 has at most one successor while the vertex with level 1 has no successor. Also no vertex of degree 2 has a successor.

The areas of the vertices are then amalgamated. Any vertex X with degree 2 or level 1 has amalgamated area x' equal to x, the area of X. Any other vertex Y has amalgamated area, y', equal to the sum of the amalgamated areas of its successors and y, the area of Y. This makes the sum of the amalgamated areas of the vertices in the initial triangle equal to the total area of all vertices in G.

Division of rectangle

Assume a rectangle, whose area equals the total area of the vertices in G, to be the plan boundary of the floorplan has been given.

Place a horizontal line across this rectangle dividing it into two so that the upper rectangle has the area of the vertex with level 1. Divide the lower rectangle with a vertical line so that the two rectangles so formed have areas equal to the amalgamated areas of the vertices with levels 2 and 3. Label these rectangles with the vertices to which they correspond.

The remaining vertices are placed in the floorplan as follows:

Step 1: Consider a room Z in the floorplan which has successors not yet placed in the floorplan. Then, as we shall show, Z is rectangular and is adjacent to two other rooms X and Y where X is rectangular and Y is either rectangular or L-shaped. Assume without loss of generality that X meets Z along a vertical wall. One of the following 4 cases occurs.

Case (i) Y is rectangular. Z has only 1 successor R required to be adjacent to either X or Y.

If R is to be adjacent to X (or Y), then place a horizontal (or vertical) line across the rectangle R (after removing its label) so that the lower (or left-most) rectangle has area equal to the amalgamated area r' of R. Label this rectangle R and the other Z.

Case (ii) Y is rectangular. Z has two successors R and S, where R is required to be adjacent to X and S to Y.

Form room R as in case (i).

Draw in the vertical line $\alpha\beta$ and the horizontal line $\beta\gamma$, where α lies on the wall between rooms Z and Y, γ is on the plan boundary and β lies within rectangle Z, across rectangle R so that the area of the L-shaped room so created is z, the area of Z. Delete the label Z, and label the L-shaped room as Z and the remaining rectangle as S. This is illustrated in figure 9.1(a).

Case (iii) Y is L-shaped. Z has only 1 successor R.

If R is required to be adjacent to Y then divide Z as in case (i). Otherwise (R adjacent to X), divide Z as in case (iii) when room S was created replacing S by R, horizontal by vertical and vice versa.

Case (iv) Y is L-shaped. Z has two successors R and S, where R is to be adjacent to X and S to Y.

Form room S as in case (i) replacing R by S. Form room R as in case (iii). This is shown in figure 9.1(b).

Should X meet Y along a horizontal wall then a similar division occurs for each of the four cases.

Step 2: If unplaced rooms still exist return to step 1, otherwise end.

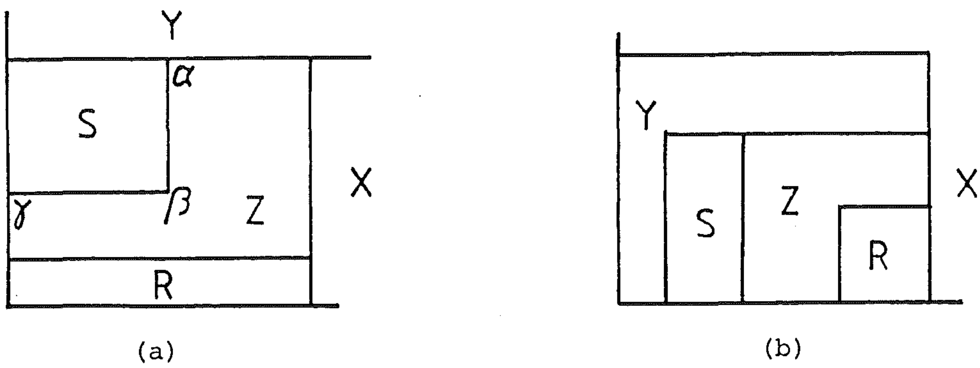


Figure 9.1 The situation for case (ii),(a) and case (iv),(b) in theorem 9.1.

It can be easily seen from the figures that each room formed in this manner is rectangular or L-shaped, has the correct area and satisfies the adjacency requirements of G. Further each successor of Z is rectangular and bears one of the four relationships to the rooms already placed in the floorplan which we have assumed in this construction. The result of the theorem follows. #

Figure 9.2(b) shows a floorplan satisfying the adjacency requirements of G in figure 9.2(a) constructed according to this theorem. Table 9.1 gives the areas, amalgamated areas, levels and successors of the vertices. C,L,M are the vertices of the initial triangle.

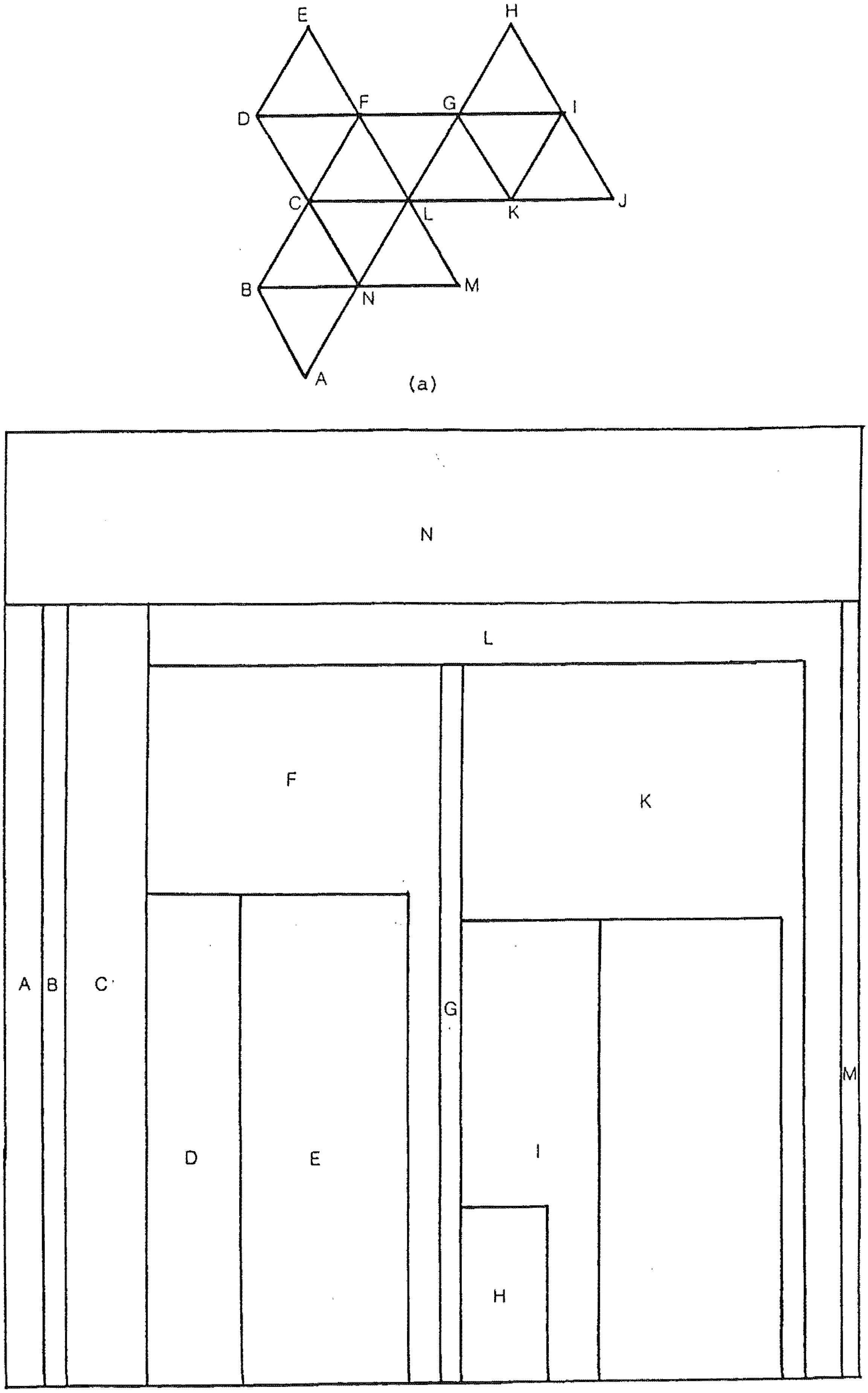


Figure 9.2 An orthogonal floorplan (b) with each room rectangular or L-shaped having the graph (a) as its weak dual and satisfying the room areas in table 9.1.

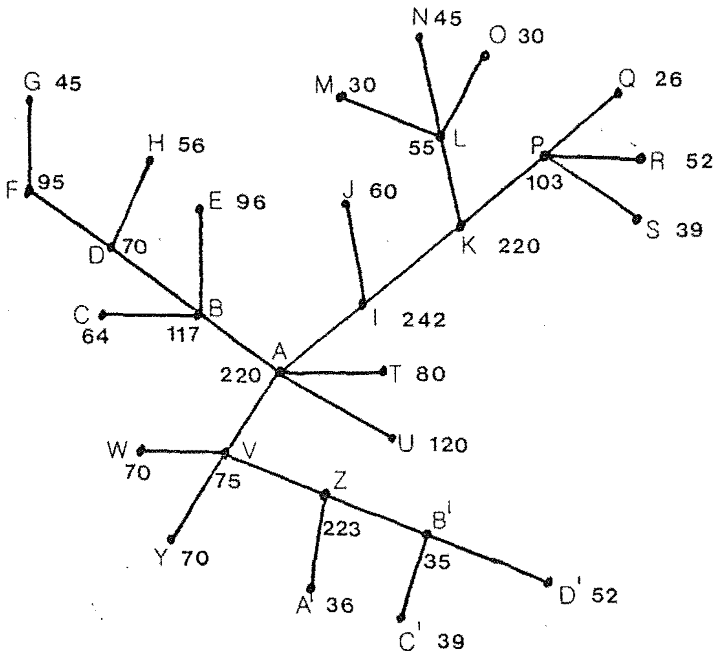
Table 9.1 Areas and other data for the vertices in figure 9.2(a)

Vertex	Area	Level	Precursor	Amalgamated Area
A	10	5	B	10
B	5	4	C	15
C	20	2		35
D	15	7	F	40
E	25	8	D	25
F	25	6	L	145
G	5	9	F	80
H	5	12	I	5
I	15	11	K	45
J	25	13	I	25
K	30	10	G	75
L	20	3		170
M	5	14	L	5
N	45	1		45
TOTAL	250			

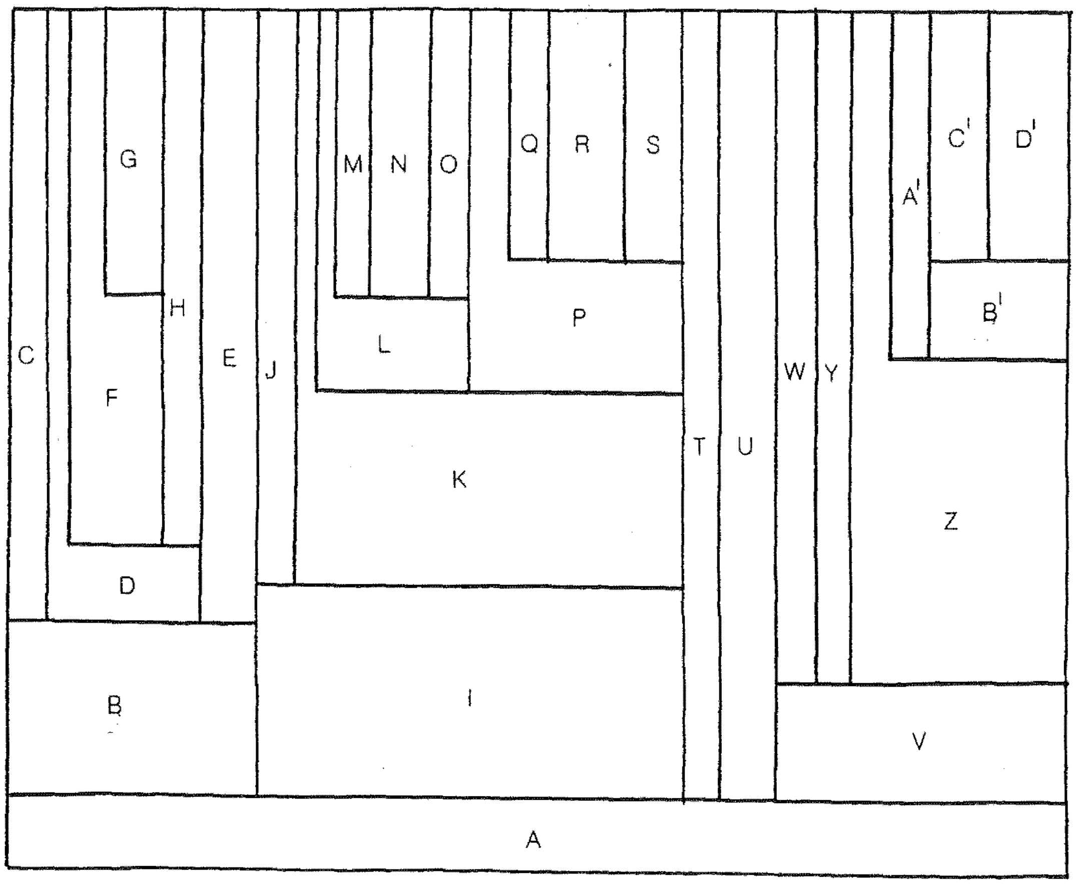
II. TREE ADJACENCY

<u>Theorem 9.2</u> An orthogonal floorplan in which each room is rectangular or L-shaped satisfying the area and adjacency requirements given by a tree T, with or without the further condition that each room is external, can always be found.

<u>Proof</u>: Since from lemma 3.22 T can be embedded in a maximal outerplanar graph G, it follows from theorem 9.1 that an orthogonal floorplan satisfying the conditions of the theorem with the stronger condition that each room is external is always possible. Without this extra condition theorem 6.2 states that a rectangular floorplan is always possible. An orthogonal floorplan is clearly then also possible.



(a)



(b)

Figure 9.3 An orthogonal floorplan (b) in which each room is external and rectangular or L-shaped satisfying the areas and adjacencies of T,(a), formed in the way outlined by theorem 9.2.

In fact, as shown in figure 9.3, an orthogonal floorplan with each room external can be drawn directly from the tree T. One Vertex A is chosen to be the root of the tree and the corresponding room becomes an endroom in the floorplan. The floorplan is then divided by vertical lines into rectangles whose areas equal the sum of the areas of the vertices in the subtrees of T-A. Each S_i, of these subtrees is taken as rooted as the vertex adjacent in T to A. The corresponding rectangle R_i is then subdivided. The root X or S_i, if X is not a terminal vertex in T and if R_i has only one wall on the plan boundary, corresponds to an L-shaped room. Otherwise room X is a rectangle. The remainder of R_i is divided as before by vertical lines into rectangles with areas equal to the sum of the vertices in the subtrees of S_i-X. Continuing in a similar fashion the floorplan is formed. #